Typing Rules for Quotient Polymorphism

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This document supplements the "Quotient Polymorphism" paper by providing the full set of typing rules for the core language λ_Q, including support for choice polymorphism for quotient types.

Key

Context: Γ, Δ, Σ	Substitution: θ	Monotype: $\tau, \tau', \tau_k, \tau'_k$	Polytype: $\sigma, \sigma', \sigma_k, \sigma'_k$
Type variable: $\alpha, \beta, \alpha_k, \beta_k$	Variable: x	Term: e, e', e_k	Refinement: ϕ, ψ
Equality constructor: ε, ε'	Quotient Set: $\mathbf{Q}$, $\mathbf{Q}$'	Quotient: Q	Constant: c
Quotient variable: q	Patterns: ρ, ρ', ρ_k	Qualifier Set: $\mathbb{Q}$	

Well-Formed Types

$$\boxed{\Gamma \vdash \sigma}$$

$$\frac{\Gamma, x : \tau \vdash \phi : \mathbf{Bool}}{\Gamma \vdash \{x : \tau \mid \phi\}} \text{[WT-BASE]} \qquad \frac{}{\Gamma \vdash \alpha} \text{[WT-VAR]}$$

$$\frac{\Gamma, x : \tau \vdash \tau'}{\Gamma \vdash (x : \tau) \to \tau'} \text{[WT-FUN]} \qquad \frac{\Gamma \vdash \sigma}{\Gamma \vdash \forall\, \alpha . \sigma} \text{[WT-POLY]}$$

$$\frac{\Gamma \vdash \tau \qquad \Gamma \vdash_{\mathbb{Q}} Q :: \sigma \qquad \Gamma \vdash \sigma \sqsubseteq \tau}{\Gamma \vdash \tau \;/\; Q} \text{[WT-QUOT]}$$

$$\frac{\Gamma, q :: \tau \vdash \sigma}{\Gamma \vdash \mathbf{choose}\ q : \tau . \sigma} \text{[WT-CHOOSE]}$$

Liquid Type Checking

$$\boxed{\Gamma \vdash_{\mathbb{Q}} e : \sigma}$$

$$\frac{\Gamma \vdash_{\mathbb{Q}} e : \sigma \qquad \Gamma \vdash \sigma \sqsubseteq \sigma' \qquad \Gamma \vdash \sigma'}{\Gamma \vdash_{\mathbb{Q}} e : \sigma'} \text{[LT-SUB]}$$

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$$\frac{x : \{\ v : \tau \mid \phi\ \} \in \Gamma}{\Gamma \vdash_{\mathbb{Q}} x : \{\ v : \tau \mid v = x\ \}}\ [\text{LT-VAR}] \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash_{\mathbb{Q}} x : \tau}\ [\text{LT-VAR}]$$

$$\frac{}{\Gamma \vdash_{\mathbb{Q}} c : \text{ty}(c)}\ [\text{LT-CONST}]$$

$$\frac{\Gamma, x : \tau \vdash_{\mathbb{Q}} e : \tau' \qquad \Gamma \vdash (x : \tau) \to \tau'}{\Gamma \vdash_{\mathbb{Q}} \lambda x.e : (x : \tau) \to \tau'}\ [\text{LT-FUN}]$$

$$\frac{\begin{array}{c}\Gamma \models (x : \tau\ /\ \mathbf{Q}) \to \tau' \qquad \Gamma \vdash_{\mathbb{Q}}: \rho_i : \tau \qquad \Gamma; \mathbf{vars}(\rho_i) \vdash_{\mathbb{Q}} e_i : \tau'[x \mapsto \rho_i] \\ \Gamma \vdash \sigma \sqsubseteq \tau \qquad \forall i \in \{1,\ \ldots,\ n\} \qquad \Gamma \models \phi \Rightarrow \lambda\ \{\ \rho \to e\ \} \rightsquigarrow \mathbf{Q} :: \sigma \\ \rho_1,\ \ldots,\ \rho_n \text{ is a complete case analysis of } \tau \qquad \text{Valid}(\llbracket\Gamma\rrbracket \Rightarrow \phi)\end{array}}{\Gamma \vdash_{\mathbb{Q}} \lambda\ \{\ \rho_1 \to e_1; \ldots; \rho_n \to e_n\ \} : (x : \tau\ /\ \mathbf{Q}) \to \tau'}\ [\text{LT-CASE}]$$

$$\frac{\Gamma \vdash_{\mathbb{Q}} f : (x : \tau) \to \tau' \qquad \Gamma \vdash (x : \tau) \to \tau'}{\Gamma \vdash_{\mathbb{Q}} \lambda x.e : (x : \tau) \to \tau'}\ [\text{LT-APP}]$$

$$\frac{\Gamma \vdash_{\mathbb{Q}} b : \text{Bool} \qquad \Gamma \vdash_{\mathbb{Q}} e : \tau \qquad \Gamma \vdash_{\mathbb{Q}} e' : \tau \qquad \Gamma \vdash \tau}{\Gamma \vdash_{\mathbb{Q}} \mathbf{if}\ b\ \mathbf{then}\ e\ \mathbf{else}\ e' : \tau}\ [\text{LT-IF}]$$

$$\frac{\Gamma \vdash_{\mathbb{Q}} e : \sigma \qquad \Gamma, e : \sigma \vdash_{\mathbb{Q}} e' : \tau \qquad \Gamma \vdash \tau}{\Gamma \vdash_{\mathbb{Q}} \mathbf{let}\ x = e\ \mathbf{in}\ e' : \tau}\ [\text{LT-LET}]$$

$$\frac{\Gamma \vdash_{\mathbb{Q}} e : \sigma \qquad \alpha \text{ not free in } \Gamma}{\Gamma \vdash_{\mathbb{Q}} e : \forall \alpha.\sigma}\ [\text{LT-GEN}]$$

$$\frac{\Gamma \vdash \tau \qquad \Gamma, q :: \tau \vdash_{\mathbb{Q}} e : \tau' \qquad q \text{ is not free in } \Gamma}{\Gamma \vdash_{\mathbb{Q}} e : \mathbf{choose}\ q : \tau.\tau'}\ [\text{LT-CHOOSE}]$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathbb{Q}} e : \mathbf{choose}\ q : \sigma_2.\sigma_3 \qquad \Gamma \vdash_{\mathbb{Q}} Q :: \sigma_1 \qquad \Gamma \vdash \sigma_1 \sqsubseteq \sigma_2 \\ \Gamma \models \phi \Rightarrow (e ::_q \sigma_3)\ \mathbf{respects}\ (Q :: \sigma_2) \qquad \text{Valid}\,(\llbracket\Gamma\rrbracket \Rightarrow \llbracket\phi\rrbracket)\end{array}}{\Gamma \vdash_{\mathbb{Q}} e : \tau'[q \mapsto Q]}\ [\text{LT-CINST}]$$

Subtyping

$\boxed{\Gamma \vdash \sigma <: \sigma'}$

$$\frac{\text{Valid}\,(\llbracket\Gamma\rrbracket \wedge \llbracket\phi\rrbracket \Rightarrow \llbracket\psi\rrbracket)}{\Gamma \vdash \{\ x : \tau \mid \phi\ \} <: \{\ x : \tau \mid \psi\ \}}\ [\text{ST-BASE}]$$

$$\frac{\Gamma \vdash \tau_2 <: \tau_1 \qquad \Gamma, x : \tau_2 \vdash \tau_1' <: \tau_2'}{\Gamma \vdash (x : \tau_1) \to \tau_1' <: (x : \tau_2) \to \tau_2'}\ [\text{ST-FUN}]$$

$$\frac{}{\Gamma \vdash \alpha <: \alpha}\ [\text{ST-VAR}] \qquad \frac{\Gamma \vdash \sigma <: \sigma'}{\Gamma \vdash \forall \alpha . \sigma <: \forall \alpha . \sigma'}\ [\text{ST-POLY}]$$

$$\frac{}{\Gamma \vdash \tau <: \tau \ / \ Q}\ [\text{ST-QBASE}] \qquad \frac{\Gamma \vdash \tau <: \tau'}{\Gamma \vdash \tau \ / \ Q <: \tau' \ / \ Q}\ [\text{ST-QUOTY}]$$

$$\frac{\Gamma; \emptyset; \emptyset \vdash \mathbf{Q} <: \mathbf{Q}'}{\Gamma \vdash \tau \ / \ \mathbf{Q} <: \tau \ / \ \mathbf{Q}'}\ [\text{ST-QUOT}] \qquad \frac{\Gamma, q :: \sigma_1 \vdash \sigma <: \sigma' \qquad \Gamma \vdash \sigma_2 \sqsubseteq \sigma_1}{\Gamma \vdash \textbf{choose}\ q : \sigma_1 . \sigma <: \textbf{choose}\ q : \sigma_2 . \sigma'}\ [\text{ST-CHOOSE}]$$

Equality constructor subtyping

$\boxed{\Gamma; \Delta; \Sigma \vdash \varepsilon <: \varepsilon'}$

$$\frac{\begin{array}{c}\Gamma, \Delta \vdash_{\mathbb{Q}} \rho, e : \tau \qquad \Gamma, \Sigma \vdash_{\mathbb{Q}} \rho', e' : \tau \qquad \rho \subseteq_{\theta} \rho' \\ \text{Valid}\,(\llbracket \Delta \rrbracket \Rightarrow \llbracket \Sigma[\theta] \rrbracket) \qquad \text{Valid}\,(\llbracket e \rrbracket = \llbracket e'[\theta] \rrbracket)\end{array}}{\Gamma; \Delta; \Sigma \models (\rho == e) <: (\rho' == e')}\ [\text{EST-BASE}]$$

$$\frac{\Gamma; \Delta, v : \tau; \Sigma \vdash_{\mathbb{Q}} \varepsilon <: \varepsilon'}{\Gamma; \Delta; \Sigma \vdash_{\mathbb{Q}} (\textbf{forall}\ v : \tau\ \textbf{in}\ \varepsilon) <: \varepsilon'}\ [\text{EST-LEFT}]$$

$$\frac{\Gamma; \Delta; \Sigma, v : \tau \vdash_{\mathbb{Q}} \varepsilon <: \varepsilon'}{\Gamma; \Delta; \Sigma \vdash_{\mathbb{Q}} \varepsilon <: (\textbf{forall}\ v : \tau\ \textbf{in}\ \varepsilon')}\ [\text{EST-RIGHT}]$$

Specialisation

$\boxed{\Gamma \vdash \sigma \sqsubseteq \sigma'}$

$$\frac{\Gamma \vdash \sigma <: \sigma'}{\Gamma \vdash \sigma \sqsubseteq \sigma'}\ [\text{SP-SUB}]$$

$$\frac{\tau' = \tau[\alpha_i \mapsto \tau_i] \qquad \beta_i \text{ not free in } \forall \alpha_1 \ \ldots \ \alpha_n . \tau}{\Gamma \vdash \forall \alpha_1 \ \ldots \ \alpha_n . \tau \sqsubseteq \forall \beta_1 \ \ldots \ \beta_m . \tau'}\ [\text{ST-POLY}]$$

Well-Formed Equality Constructors

$\boxed{\Gamma; \Delta \vdash_{\mathbb{Q}} \varepsilon :: \tau}$

$$\frac{\Gamma; \Delta, v : \tau' \vdash_{\mathbb{Q}} \varepsilon :: \tau}{\Gamma; \Delta \vdash_{\mathbb{Q}} \textbf{forall}\ v : \tau'\ \textbf{in}\ \varepsilon :: \tau}\ [\text{EC-BIND}] \qquad \frac{\Gamma, \Delta \vdash_{\mathbb{Q}} \rho : \tau \qquad \Gamma, \Delta \vdash_{\mathbb{Q}} e : \tau}{\Gamma; \Delta \vdash_{\mathbb{Q}} \rho == e :: \tau}\ [\text{EC-BASE}]$$

Well-Formed Quotients

$\boxed{\Gamma \models Q :: \sigma}$

$$\frac{\Gamma; \emptyset \vdash_{\mathbb{Q}} \varepsilon_i :: \tau \qquad \Gamma \vdash \sigma \sqsubseteq \tau \qquad \forall\, i \in \{1, \ldots, n\}}{\Gamma \vdash_{\mathbb{Q}} \{e_i \mid i \in \{1, \ldots, n\}\} :: \sigma}\ [\text{QT-BASE}]$$

$$\frac{\Gamma \vdash_{\mathbb{Q}} Q :: \sigma \qquad \alpha \text{ free in } \Gamma}{\Gamma \vdash_{\mathbb{Q}} Q :: \forall \alpha . \sigma} \text{[QT-POLY]} \qquad \frac{q :: \sigma \in \Gamma}{\Gamma \vdash_{\mathbb{Q}} q :: \sigma} \text{[QT-VAR]}$$

Equality Constructor Respectfulness $\boxed{\Gamma; \Delta \models \phi \Rightarrow \lambda \{ \rho \to e \} \rightsquigarrow \varepsilon :: \tau}$

$$\frac{\Gamma, \Delta \vdash_{\mathbb{Q}} \rho' : \tau \qquad \Gamma, \Delta \vdash_{\mathbb{Q}} e' : \tau \qquad \exists\, i \in \{1, \ \ldots, \ n\}.\ \rho_i \sim_{\theta} \rho'}{\Gamma; \Delta \models (\llbracket \Delta[\theta] \rrbracket,\ e_i[\theta],\ \lambda \{ \rho_1 \to e_1;\ \ldots;\ \rho_n \to e_n \}\ e'[\theta]) \Rightarrow \lambda \{ \rho \to e \} \rightsquigarrow (\rho' == e') :: \tau} \text{[RP-BASE]}$$

$$\frac{\Gamma; \Delta, v : \tau' \models \phi \Rightarrow (\rho \to e) \rightsquigarrow \varepsilon :: \tau}{\Gamma; \Delta \models \phi \Rightarrow \lambda \{ \rho \to e \} \rightsquigarrow (\textbf{forall}\ v : \tau'\ \textbf{in}\ \varepsilon) :: \tau} \text{[RP-BIND]}$$

Quotient Respectfulness $\boxed{\Gamma \models \phi \Rightarrow \lambda \{ \rho \to e \} \rightsquigarrow \mathbf{Q} :: \tau}$

$$\frac{\Gamma \vdash_{\mathbb{Q}} \mathbf{Q} :: \sigma \qquad \Gamma \vdash \sigma \sqsubseteq \tau}{\Gamma \models \{ \psi \mid \varepsilon \in \mathbf{Q},\ \Gamma; \emptyset \models \psi \Rightarrow (\rho \to e) \rightsquigarrow \varepsilon :: \tau \} \Rightarrow \lambda \{ \rho \to e \} \rightsquigarrow \mathbf{Q} :: \tau} \text{[RP-QUOT]}$$

Polymorphic Respectfulness $\boxed{\Gamma; \Delta \models \phi \Rightarrow (e ::_q \sigma)\ \textbf{respects}\ (\varepsilon :: \sigma')}$

$$\frac{\Gamma; \Delta, x : \tau_1 \models \phi \Rightarrow (e ::_q \tau_2)\ \textbf{respects}\ (\varepsilon :: \tau_3)}{\Gamma; \Delta \models \phi \Rightarrow (e ::_q \tau_2)\ \textbf{respects}\ (\textbf{forall}\ x : \tau_1\ \textbf{in}\ \varepsilon :: \tau_3)} \text{[PR-BIND]}$$

$$\frac{}{\Gamma; \Delta \models \emptyset \Rightarrow (c ::_q \mathrm{ty}(c))\ \textbf{respects}\ (\rho == e :: \sigma)} \text{[PR-CONST]}$$

$$\frac{x : \sigma \in \Gamma}{\Gamma; \Delta \models \emptyset \Rightarrow (x ::_q \sigma)\ \textbf{respects}\ (\rho == e :: \sigma')} \text{[PR-VAR]}$$

$$\frac{\begin{array}{c}\Gamma \vdash \sigma \sqsubseteq \tau \qquad \forall\, i \in \{1, \ldots, n\} \qquad \Gamma \vdash_{\mathbb{Q}} \rho_i : \tau \ /\ q \\ \Gamma; \Delta \models \psi \Rightarrow \lambda \{ \rho \to e \} \rightsquigarrow \rho == e :: \sigma \\ \Gamma, \textbf{vars}(\rho_i); \Delta \models \phi_i \Rightarrow (e_i ::_q \tau'[x \mapsto \rho_i])\ \textbf{respects}\ (\rho == e :: \sigma)\end{array}}{\Gamma; \Delta \models (\bigcup_i \llbracket \tau \rrbracket * \phi_i) \cup \psi \Rightarrow (\lambda \{\rho_i \to e_i\} ::_q (x : \tau \ /\ q) \to \tau')\ \textbf{respects}\ (\rho == e :: \sigma)} \text{[PR-CASE]}$$

$$\frac{\begin{array}{c}\Gamma \vdash \sigma \sqsubseteq \tau \qquad \forall\, i \in \{1, \ldots, n\} \qquad \Gamma \vdash_{\mathbb{Q}} \rho_i : \tau \ /\ Q \qquad Q \neq q \\ \Gamma, \textbf{vars}(\rho_i); \Delta \models \phi_i \Rightarrow (e_i ::_q \tau'[x \mapsto \rho_i])\ \textbf{respects}\ (\rho == e :: \sigma)\end{array}}{\Gamma; \Delta \models \bigcup_i \llbracket \tau \rrbracket * \phi_i \Rightarrow (\lambda \{\rho_i \to e_i\} ::_q (x : \tau \ /\ Q) \to \tau')\ \textbf{respects}\ (\rho == e :: \sigma)} \text{[PR-NCASE]}$$

$$\frac{\Gamma, x : \tau; \Delta \models \phi \Rightarrow (e ::_q \tau') \textbf{ respects } (\rho == e :: \sigma)}{\Gamma; \Delta \models [\![\tau]\!] * \phi \Rightarrow (\lambda\, x.e ::_q (x : \tau) \to \tau') \textbf{ respects } (\rho == e :: \sigma)} \text{ [PR-FUN]}$$

$$\frac{\begin{array}{c}\Gamma; \Delta \models \phi \Rightarrow (f ::_q (x : \tau) \to \tau') \textbf{ respects } (\rho == e :: \sigma) \\ \Gamma; \Delta \models \psi \Rightarrow (e ::_q \tau) \textbf{ respects } (\rho == e :: \sigma)\end{array}}{\Gamma; \Delta \models \phi\ e \cup f\ \psi \Rightarrow (f\ e ::_q \tau'[x \mapsto e]) \textbf{ respects } (\rho == e :: \sigma)} \text{ [PR-APP]}$$

$$\frac{\begin{array}{c}\Gamma \vdash_{\mathbb{Q}} b : \textbf{Bool} \qquad \Gamma; \Delta \models \phi \Rightarrow (e ::_q \tau) \textbf{ respects } (\rho == e :: \sigma) \\ \Gamma; \Delta \models \psi \Rightarrow (e' ::_q \tau) \textbf{ respects } (\rho == e :: \sigma)\end{array}}{\Gamma; \Delta \models [\![b]\!] * \phi \cup \neg[\![b]\!] * \psi \Rightarrow (\textbf{if } b \textbf{ then } e \textbf{ else } e' ::_q \tau) \textbf{ respects } (\rho == e :: \sigma)} \text{ [PR-IF]}$$

$$\frac{\Gamma, x : \sigma; \Delta \models \phi \Rightarrow (e' ::_q \tau) \textbf{ respects } (\rho == e :: \sigma)}{\Gamma; \Delta \models \phi[x \mapsto e] \Rightarrow (\textbf{let } x = e \textbf{ in } e' ::_q \tau') \textbf{ respects } (\rho == e :: \sigma)} \text{ [PR-LET]}$$

Polymorphic Quotient Respectfulness $\boxed{\Gamma \models \phi \Rightarrow (e ::_q \sigma) \textbf{ respects } (Q :: \sigma')}$

$$\frac{r :: \sigma_2 \in \Gamma \qquad \Gamma \vdash \sigma_2 \sqsubseteq \sigma_2'}{\Gamma \models \emptyset \Rightarrow (e ::_q \sigma_1) \textbf{ respects } (r :: \sigma_2')} \text{ [PR-QVAR]}$$

$$\frac{\Gamma; \emptyset \models \phi_i \Rightarrow (e ::_q \sigma) \textbf{ respects } (\varepsilon_i :: \tau) \qquad \forall\, i \in \{1, \ldots, n\}}{\Gamma \models \bigcup_i \phi_i \Rightarrow (e ::_q \sigma) \textbf{ respects } (\{\ \varepsilon_i \mid i \in \{1, \ldots, n\}\ \} :: \sigma')} \text{ [PR-QUOT]}$$