The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN 2009–2010

ALGORITHMIC PROBLEM SOLVING

Time allowed 1 hour 30 minutes

Candidates must NOT start writing their answers until told to do so.

Model Solutions

Question 1:

a)

Assignment	Expression	Invariant?
$n \;:=\; n{+}1$	$n \bmod 3$	No, $0 \bmod 3 \neq 1 \bmod 3$
$n \;:=\; n{+}21$	$n \bmod 7$	Yes
$m\,,n \;:=\; m{+}1\,,n{-}2$	$6{\times}m+3{\times}n$	Yes
$m\,,n \;:=\; m{+}1\,,n{-}2$	$3{\times}m+6{\times}n$	No, $3{\times}1+6{\times}2 \neq 3{\times}2+6{\times}0$
$d\,,e \;:=\; d\,,\neg e$	$d=e$	No, $(false=true)\neq(false=false)$

Table 0.1 Fill in entries marked "?"

The second expression is an invariant of the assignment, since:

$$
\begin{array}{ll}
 & (n \bmod 7)[n \;:=\; n{+}21] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & (n{+}21) \bmod 7 \\
= & \{ \quad 21 \text{ is a multiple of } 7 \quad \} \\
 & n \bmod 7 \quad .
\end{array}
$$

The third expression is an invariant of the assignment, since:

$$
\begin{array}{ll}
 & (6{\times}m+3{\times}n)[m\,,n \;:=\; m{+}1\,,n{-}2] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & 6{\times}(m{+}1)+3{\times}(n{-}2) \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & 6{\times}m+6+3{\times}n-6 \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & 6{\times}m+3{\times}n \quad .
\end{array}
$$

b) (i) $pg \equiv \neg ps$ (ii) $ig \equiv (ig \equiv \neg is)$ (iii) $is \equiv (is \Leftarrow pg)$

(iv) The following calculation shows that the portrait is in the gold casket. The inscription on the silver casket is false. We can't say anything about the inscription on the gold casket.

$$
\begin{array}{ll}
 & (pg \equiv \neg ps) \wedge (ig \equiv (ig \equiv \neg is)) \wedge (is \equiv (is \Leftarrow pg)) \\
= & \{ \quad \text{associativity and negation} \quad \} \\
 & (pg \equiv ps \equiv false) \wedge ((ig \equiv ig) \equiv is \equiv false) \wedge (is \equiv (is \Leftarrow pg)) \\
= & \{ \quad \text{reflexivity} \quad \} \\
 & (pg \equiv ps \equiv false) \wedge (is \equiv false) \wedge (is \equiv (is \Leftarrow pg)) \\
= & \{ \quad \text{definition of } \Leftarrow \quad \} \\
 & (pg \equiv ps \equiv false) \wedge (is \equiv false) \wedge (is \equiv is \equiv is \vee pg) \\
= & \{ \quad \text{substitution of equals for equals, reflexivity} \quad \} \\
 & (pg \equiv ps \equiv false) \wedge (is \equiv false) \wedge (true \equiv pg)
\end{array}
$$

$= \quad \{ \quad$ substitution of equals for equals, reflexivity $\quad \}$

$(ps \equiv false) \wedge (is \equiv false) \wedge (true \equiv pg)$.

(v) The following calculation shows that the portrait is in the silver casket. The inscription on the silver casket is false. We can't say anything about the inscription on the gold casket.

$(pg \equiv \neg ps) \wedge (ig \equiv (ig \equiv \neg is)) \wedge (is \equiv pg)$

$= \quad \{ \quad$ associativity and negation $\quad \}$

$(pg \equiv ps \equiv false) \wedge ((ig \equiv ig) \equiv is \equiv false) \wedge (is \equiv pg)$

$= \quad \{ \quad$ reflexivity, substitution of equals for equals $\quad \}$

$(pg \equiv ps \equiv false) \wedge (pg \equiv false) \wedge (is \equiv pg)$

$= \quad \{ \quad$ substitution of equals for equals, reflexivity $\quad \}$

$(ps \equiv true) \wedge (pg \equiv false) \wedge (is \equiv false)$

Question 2: a) see fig. 0.1.

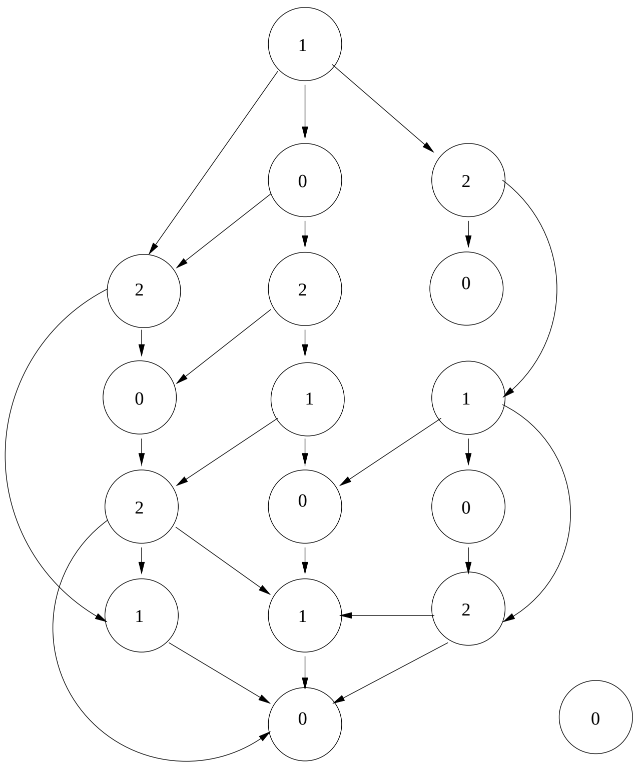

Figure 0.1 Question 2. Mex Numbers.

b)

Left Game	Right Game	"losing" or winning move
Q	7	losing
K	17	1
B	9	losing
N	32	2
D	9	A or 2

Table 0.2 Fill in entries marked "?"

Question 3:

a) (i) The assignment is:

$$g\,,b\,,c \quad := \quad g{-}1\,,b{+}2\,,c{-}1$$

(ii) The expression $2{\times}g+b$ is a valid example.

(iii) The expression that denotes the total number of chameleons is $g{+}b{+}c$. We prove that it is an invariant of the assignment as follows:

$$(g{+}b{+}c)[g\,,b\,,c \quad := \quad g{-}1\,,b{+}2\,,c{-}1]$$

= { assignment rule }

$$(g{-}1){+}(b{+}2){+}(c{-}1)$$

= { arithmetic }

$$g{+}b{+}c \;.$$

b) (This question is derived from the courseworks.)

Two strategies for taking the two slowest across are to take them separately, using the fastest to carry the torch, or to let them cross together, using the two fastest to return the torch to the initial bank. In this case, it is better to let them cross together. This gives the sequence: $\langle p_1,p_5\rangle, [p_1]\,,\langle p_{15},p_{20}\rangle, [p_5]$ (Angle brackets indicate a crossing in one direction, square brackets a crossing in the other direction.) After this sequence, just the two slowest have reached the opposite bank, and the torch is at the initial bank. Now the two slowest are p_8 and p_{10} . For these, the optimal strategy is to let them cross separately, with the fastest carrying the torch. This gives the sequence: $\langle p_1,p_8\rangle, [p_1]\,,\langle p_1,p_{10}\rangle, [p_1]$ The process is completed by p_1 and p_5 crossing together: $\langle p_1,p_5\rangle$.

The total time taken is $5{+}1{+}20{+}5{+}8{+}1{+}10{+}1{+}5$, i.e. 56 minutes.

Question 4:

a) (i) Consider the two closest neighbours A and B (the pair of closest neighbours is unique, since all distances are different). Because there is no one closer to A than B and no one closer to B than A, they must shoot each other. This leaves the third person dry.

(ii) Assume that for an odd number n of people, suitably located so that, for each of them, all the others are at a different distance away, we have that one lucky person does not get wet after the firing signal. This is the induction hypothesis.

We now have to prove that the property holds for $n+2$ people (remember that n is odd). Consider the two closest neighbours A and B (the pair of closest neighbours is unique, since all distances are different). Because there is no one closer to A than B and no one closer to B than A, they must shoot each other.

This leaves us with n people, where n is odd and for each person, each of the others is still a different distance away. Therefore, the induction hypothesis implies that there is some person X who does not get wet.

Note that the presence of A and B, who shoot each other, would not induce anyone to shoot X; if anything, the only effect of introducing A and B would be to draw off the fire of those members who happen to have A or B as their nearest neighbour. In any case, X remains dry.

(Since the claim holds for $n=3$ (see part (i)), the conclusion follows by induction for all odd n.)

b)

(i) $N1 \not\equiv O2$ (ii) $N2 \not\equiv O3$ (iii) $O4 \not\equiv L2$

(iv) The following calculation shows how to deduce the final positions.

$$\begin{array}{ll}
 & (N1 \not\equiv O2) \wedge (N2 \not\equiv O3) \wedge (O4 \not\equiv L2) \\
= & \{ \quad \text{conjunction distributes over inequivales} \quad \} \\
 & ((N1 \not\equiv O2) \wedge N2 \;\;\not\equiv\;\; (N1 \not\equiv O2) \wedge O3) \;\;\wedge\;\; (O4 \not\equiv L2) \\
= & \{ \quad \text{conjunction distributes over inequivales} \quad \} \\
 & (N1 \wedge N2 \;\;\not\equiv\;\; O2 \wedge N2 \;\;\not\equiv\;\; N1 \wedge O3 \;\;\not\equiv\;\; O2 \wedge O3) \wedge (O4 \not\equiv L2) \\
= & \{ \quad pm \wedge pn \equiv m = n \text{ and } pn \wedge qn \equiv p = q \quad \} \\
 & (false \;\;\not\equiv\;\; false \;\;\not\equiv\;\; N1 \wedge O3 \;\;\not\equiv\;\; false) \wedge (O4 \not\equiv L2) \\
= & \{ \quad false \text{ is the unit of inequivales} \quad \} \\
 & (N1 \wedge O3) \wedge (O4 \not\equiv L2) \\
= & \{ \quad \text{conjunction distributes over inequivales} \quad \} \\
 & N1 \wedge O3 \wedge O4 \not\equiv N1 \wedge O3 \wedge L2 \\
= & \{ \quad pm \wedge pn \equiv m = n \text{ and } false \text{ is the unit of inequivales} \quad \} \\
 & N1 \wedge O3 \wedge L2 \quad .
\end{array}$$

So, according to the calculation, Nancy won the race, Lucy was second, and Opey was third. Since there are four girls, we can also deduce that Minnie was last.