The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN SEMESTER 2010–2011

ALGORITHMIC PROBLEM SOLVING

Time allowed 2 hours

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

Answer FOUR questions out of SIX.

Marks available for sections of questions are shown in brackets in the right-hand margin.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject-specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: Appendix with relevant laws of Boolean algebra.

INFORMATION FOR INVIGILATORS: The final page (containing fig. 6.1) should be handed in with the examination book at the end of the examination. Please ensure that both are securely tied together.

1 a) Table 1.1 shows a list of expressions and assignments to the variables in the expressions. State for each, whether or not the expression is an invariant of the assignment. Justify your answer, either by using the invariant rule for assignment statements, or by giving a counterexample. Assume that m , n and l are integers, and d and e are booleans. **(10)**

Assignment	Expression	Invariant?
$n \;:=\; n{+}2$	n mod 5	?
$n \;:=\; n{+}81$	n mod 9	?
$m\,,n \;:=\; m{+}3\,,n{-}5$	$5{\times}m+3{\times}n$	?
$m\,,n \;:=\; m{+}3\,,n{-}5$	$5{\times}m+3{\times}n+3{\times}l$	?
$d\,,e \;:=\; \neg d\,,e$	$\neg d = \neg e$	?

Table 1.1 Fill in entries marked "?"

b) A bubble chamber contains three types of subatomic particles: particles of type X, particles of type Y, and particles of type Z. Whenever an X- and Y-particle collide, they both become Z-particles. Likewise, Y- and Z-particles collide and become X-particles; and X- and Z-particles become Y-particles upon collision.

Suppose x denotes the number of X-particles, y denotes the number of Y-particles, and z denotes the number of Z-particles.

(i) What is the simultaneous assignment that models a collision between one X-particle and one Z-particle? **(5)**

(ii) Give an expression depending on both x and y but not z that is an invariant of the assignment in part (i). **(5)**

(iii) Prove that, after one collision between one X-particle and one Z-particle, the total number of particles does not change. (Full marks can only be obtained by explicitly stating the expression that represents the total number of particles and by using the assignment rule.) **(5)**

2 Full marks for the following question can only be obtained by using the laws of Boolean algebra given in the appendix and *not* using case analysis.

In one undocumented account of the famous detective's adventures, Dr. Watson narrates the story of Holmes on the trail of his archnemesis, Prof. Moriarty. Holmes had entered a tunnel and found two pathways. Both pathways, left and right, had a note left on the ground. Presumably, Prof. Moriarty had written those notes to confuse Holmes and put him off the trail. Each note could be either true or false, but on the basis of the notes, Holmes was to choose the correct pathway and be back on the trail for his archnemesis.

The notes are:

Right: Exactly one of these notes is true.
Left: This note is true if Moriarty takes the right pathway.

Let pr stand for "Moriarty takes the right pathway", let pl stand for "Moriarty takes the left pathway", let ir stand for "the note retrieved for the right pathway is true" and let il stand for "the note retrieved for the left pathway is true".

a) What relation between pr and pl do you deduce from the fact that Prof. Moriarty had only walked taken one pathway?

(5)

b) What relation between ir and il do you deduce from the note retrieved for the right pathway?

(5)

c) What relation between il and pr do you deduce from the note retrieved for the left pathway? **(5)**

d) By combining your answers to these three questions, deduce which pathway Prof. Moriarty has taken. What can you conclude about the two notes? **(5)**

e) Suppose now that the notes retrieved for the two pathways are the following:

Right: Exactly one of these notes is true.
Left: Moriarty takes the right pathway.

Deduce which pathway Prof. Moriarty has taken. What can you conclude about the two notes?

(5)

3 a) Figure 6.1 (at the end of this examination paper) depicts the possible moves in a two-person game. The nodes represent positions in the game, and the edges represent moves. Players take it in turns to make a move; a player who is unable to move loses.

Annotate each node of fig. 6.1 with the mex number of the position. **(10)**

(Add your student identity number at the top of the page. Hand in the annotated copy of the figure securely fastened to your examination book.)

b) Consider a matchstick game with one pile of matches. Suppose a move is to remove 1, 4 or 5 matches. Construct a table showing the mex number of each position such that the number of matches is at most 15. **(8)**

c) Consider a game which is the sum of two games. The left game is the one shown in fig. 6.1. In the right game, 1, 4 or 5 matches may be removed from a pile of matches. In the sum game, a move is made by choosing to play in the left game, or choosing to play in the right game. The game ends when it is not possible to move in either game; the player whose turn it is to play loses.

The following table shows a number of different positions in this game. A position is given by a pair: the label of a node in the left game, and the number of matches in the right game. Complete the table by filling in, for each position, the word "losing" if the position is a losing position or, in the case that the position is a winning position, either the label of one of the nodes in fig. 6.1, or the number 1, the number 4 or the number 5. If a label is entered, it means move to the node with that label in the left game, and if a number is entered it means remove that number of matches in the right game. (For example, "N" means move from the current position in the left game to the position labelled "N"; "1" means remove 1 match from the pile of matches.) **(7)**

Left Game	Right Game	"losing" or winning move
G	7	?
E	36	?
C	25	?
I	19	?
M	20	?

Table 3.1 Fill in entries marked "?"

4 a) A fuse, when lit at one end, burns for a specified period of time. For example, a 1-minute fuse will burn exactly one minute.

The rate at which a fuse burns is irregular. If an m-minute fuse is cut into two pieces, one of the pieces will burn for n minutes and the other for $m-n$ minutes but, in general, there is no way of determining n. However, if an m-minute fuse is lit at both ends, it will burn for n-minutes where $n = m - n$. That is, it will burn for $\frac{1}{2}m$ minutes.

Suppose you are given one m-minute fuse and one n-minute fuse. Show how you would construct clocks to measure the following intervals of time.

(i) $m + \frac{1}{2}n$ minutes. **(3)**

(ii) $\frac{1}{2}(m+n)$ minutes. **(3)**

(iii) $m - \frac{1}{2}n$ minutes. **(3)**

(iv) $\frac{1}{2}(m - \frac{1}{2}n)$ minutes. **(3)**

(v) $\frac{1}{2}n + \frac{1}{2}(m - \frac{1}{2}n)$ minutes. **(3)**

b) Suppose you are given one 28-minute fuse and one 32-minute fuse. List the different times that you can clock using one or both fuses. Give for each one a mathematical formula that summarises how the clock is constructed. (Hint: 5 examples have already been given in part (a).) You are not required to prove that you have listed the maximum number of clocks.

(10)

5 Four couples, each consisting of a president and a bodyguard, want to cross a river. They have one rowing boat which can carry at most 3 people. Everyone can row. The presidents are afraid of the bodyguards so that none of them wishes to be with another president's bodyguard (either on a bank or in the boat) if their own bodyguard is not present.

A *state* is a combination of the position of the boat (on the left or right bank of the river) and a partitioning of the presidents and bodyguards into those on the left bank and those on the right bank. A *transition* is a partitioning of the presidents and bodyguards into those on the left bank, those on the right bank and those in the boat. States and transitions are said to be *valid* if they obey the rule that if a president is in the same place (boat or river bank) as a bodyguard then the president's own bodyguard is also in the same place.

a) List all the valid states (for the four-couple problem). You should not name individual presidents and bodyguards: give only sufficient information to deduce how many couples are on each bank and how many individual bodyguards or presidents. **(10)**

b) Construct a state-transition diagram that shows all possible ways of getting the four couples across the river. Be sure to make clear the symmetry between the left and right bank in your state-transition diagram. **(10)**

c) Using your state-transition diagram, calculate the number of different ways to get all 4 couples across the river with the least number of crossings. Explain your calculation. **(5)**

6 A group of $2 \times n$ people wish to cross a bridge. It is dark, and it is necessary to use a torch when crossing the bridge, but they only have one torch between them. The bridge is narrow and only two people can be on it at any one time. Each person takes a different amount of time to cross the bridge; when two people cross together they must proceed at the speed of the slowest. The people are all numbered from 1 to $2 \times n$, and person i takes t_i minutes to cross. The torch must be ferried back and forth across the bridge, so that it is always carried when the bridge is crossed.

You may assume that the people are ordered according to the time they take to cross, with person 1 being the fastest and person $2 \times n$ being the slowest. You may also assume that n is at least 1 (so there are at least two people).

a) Describe an algorithm based on induction on n, for a group of people wanting to cross the bridge in the shortest possible time. What is the base case and what is the induction hypothesis? (Hints: if you count the number of return trips, you will observe that there are at least two people that never return. Which people should never return?) Note that you are not required to prove that the algorithm results in the optimal crossing time.

(10)

b) Use the algorithm described in a) to solve the problem with six people, who individually cross in 1, 5, 6, 14, 16 and 20 minutes. (You may use the notation p_t for the person who takes time t to cross. So, p_1 is the one who takes 1 minute to cross, p_5 is the person who takes 5 minutes to cross, etc.) Full marks can only be obtained by giving an algorithm that minimises the total time taken.

(10)

c) What is the minimum total time taken if in addition to the six people in (b), two more people who individually cross in 25 and 30 minutes join the original group. Your answer should be based on the use of induction and results obtained earlier in (b).

(5)

STUDENT IDENTIFICATION NUMBER:

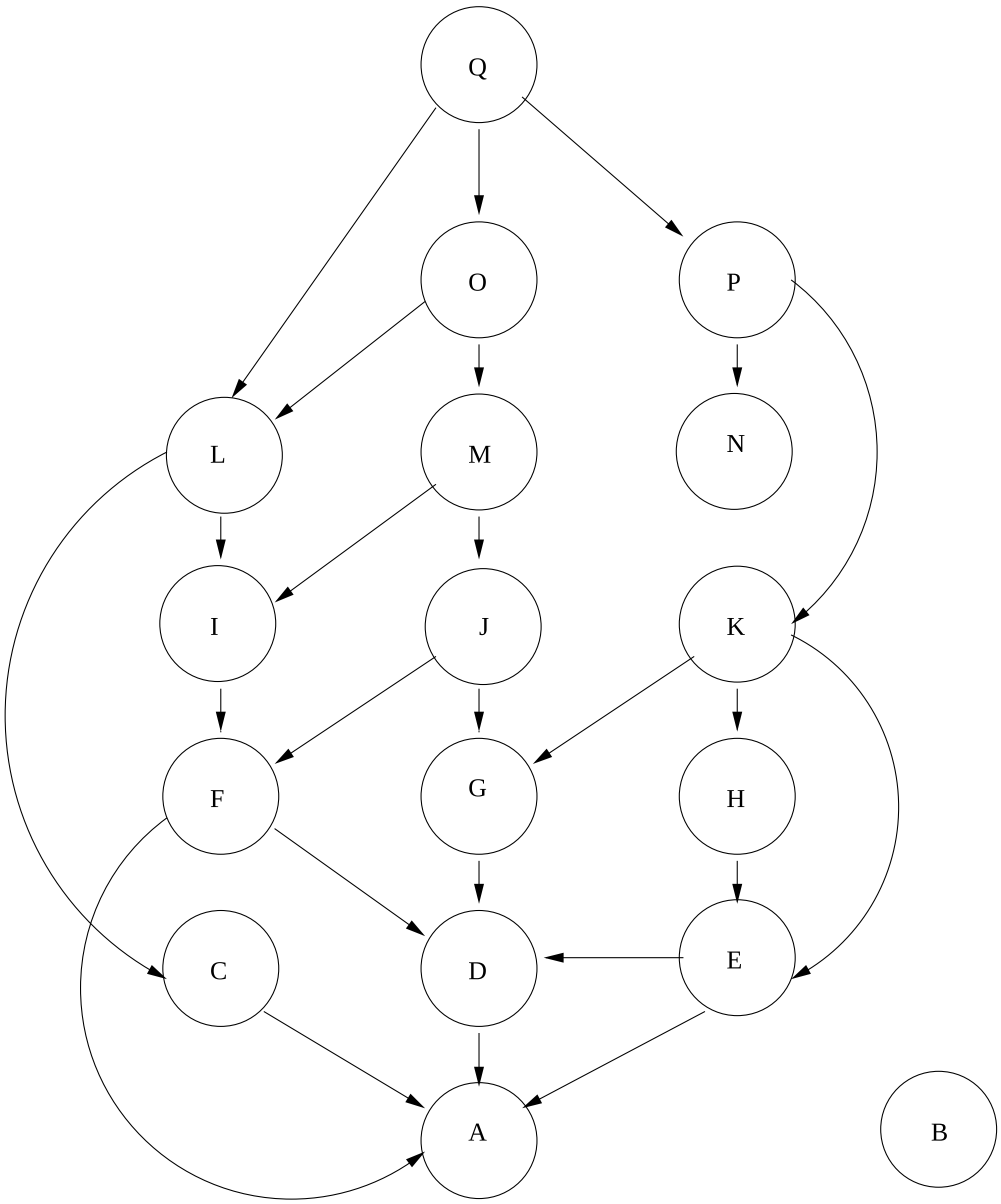

Figure 6.1: (Question 3) Label each position with its mex number. This sheet should be handed in with your examination book.

APPENDIX

Relevant Laws of Boolean Algebra

Equality of booleans is denoted by the symbol "$\equiv$". Note that, in the symmetry, unit and negation laws, the associativity of boolean equality is assumed. That is, continued expressions are read associatively and not conjunctionally. The rules are typically used in combination with Leibniz's rule (also known as "substitution of equals for equals").

associativity: $[\ ((p \equiv q) \equiv r)\ \equiv\ (p \equiv (q \equiv r))\ ]$

symmetry: $[\ p \equiv q \equiv q \equiv p\ ]$

unit: $[\ \mathsf{true} \equiv p \equiv p\ ]$

negation: $[\ \neg p \equiv p \equiv \mathsf{false}\ ]$

inequivalence: $[\ (p \not\equiv q)\ \equiv\ \neg(p \equiv q)\ ]$

if: $[\ \mathsf{true} \Leftarrow q\ \equiv\ \mathsf{true}\ ]$

if: $[\ \mathsf{false} \Leftarrow q\ \equiv\ \neg q\ ]$