The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN SEMESTER 2010–2011

ALGORITHMIC PROBLEM SOLVING

Time allowed 2 hours

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

Model Solutions

Question 1:

a)

Assignment	Expression	Invariant?
$n \;:=\; n{+}2$	$n \bmod 5$	No, $0 \bmod 5 \neq 2 \bmod 5$
$n \;:=\; n{+}81$	$n \bmod 9$	Yes
$m\,,n \;:=\; m{+}3\,,n{-}5$	$5{\times}m + 3{\times}n$	Yes
$m\,,n \;:=\; m{+}3\,,n{-}5$	$5{\times}m + 3{\times}n + 3{\times}l$	Yes
$d\,,e \;:=\; \neg d\,,e$	$\neg d = \neg e$	No, $(false = false) \neq (true = false)$

Table 0.1 Fill in entries marked "?"

The second expression is an invariant of the assignment, since:

$$(n \bmod 9)[n \;:=\; n{+}81]$$
$$= \quad \{ \quad \text{assignment rule} \quad \}$$
$$(n{+}81) \bmod 9$$
$$= \quad \{ \quad 81 \text{ is a multiple of } 9 \quad \}$$
$$n \bmod 9 \quad .$$

The third expression is an invariant of the assignment, since:

$$(5{\times}m + 3{\times}n)[m\,,n \;:=\; m{+}3\,,n{-}5]$$
$$= \quad \{ \quad \text{assignment rule} \quad \}$$
$$5{\times}(m{+}3) + 3{\times}(n{-}5)$$
$$= \quad \{ \quad \text{arithmetic} \quad \}$$
$$5{\times}m + 15 + 3{\times}n - 15$$
$$= \quad \{ \quad \text{arithmetic} \quad \}$$
$$5{\times}m + 3{\times}n \quad .$$

The fourth expression is an invariant of the assignment, since:

$$(5{\times}m + 3{\times}n + 3{\times}l)[m\,,n \;:=\; m{+}3\,,n{-}5]$$
$$= \quad \{ \quad \text{assignment rule} \quad \}$$
$$5{\times}(m{+}3) + 3{\times}(n{-}5) + 3{\times}l$$
$$= \quad \{ \quad \text{arithmetic} \quad \}$$
$$5{\times}m + 15 + 3{\times}n - 15 + 3{\times}l$$
$$= \quad \{ \quad \text{arithmetic} \quad \}$$
$$5{\times}m + 3{\times}n + 3{\times}l \quad .$$

b)i) $x\,,y\,,z \;:=\; x{-}1\,,y{+}2\,,z{-}1$

ii) $2{\times}x + y$

iii) The total is $x{+}y{+}z$. It is invariant because

$$(x{+}y{+}z)[x\,,y\,,z \;:=\; x{-}1\,,y{+}2\,,z{-}1]$$

$$
\begin{array}{ll}
= & \{ \quad \text{substitution rule} \quad \} \\
 & (x{-}1){+}y{+}2{+}(z{-}1) \\
= & \{ \quad \text{associativity and symmetry of addition} \quad \} \\
 & x{+}y{+}z{+}2{-}1{-}1 \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & x{+}y{+}z \quad .
\end{array}
$$

Question 2:

(a) $pr \equiv \neg pl$ (b) $ir \equiv (ir \equiv \neg il)$ (c) $il \equiv (il \Leftarrow pr)$

(d) The following calculation shows that Prof. Moriarty took the right pathway. The note found for the left pathway is false. We can't say anything about the note found for the right pathway.

$$
\begin{array}{ll}
 & (pr \equiv \neg pl) \wedge (ir \equiv (ir \equiv \neg il)) \wedge (il \equiv (il \Leftarrow pr)) \\
= & \{ \quad \text{associativity and negation} \quad \} \\
 & (pr \equiv pl \equiv false) \wedge ((ir \equiv ir) \equiv il \equiv false) \wedge (il \equiv (il \Leftarrow pr)) \\
= & \{ \quad \text{reflexivity twice} \quad \} \\
 & (pr \equiv pl \equiv false) \wedge (il \equiv false) \wedge (il \equiv (il \Leftarrow pr)) \\
= & \{ \quad \text{substitution of equals for equals} \quad \} \\
 & (pr \equiv pl \equiv false) \wedge (il \equiv false) \wedge (false \equiv (false \Leftarrow pr)) \\
= & \{ \quad \text{follows-from, reflexivity} \quad \} \\
 & (pr \equiv pl \equiv false) \wedge (il \equiv false) \wedge (true \equiv pr) \\
= & \{ \quad \text{substitution of equals for equals, reflexivity} \quad \} \\
 & (pl \equiv false) \wedge (il \equiv false) \wedge (true \equiv pr) \quad .
\end{array}
$$

(e) The following calculation shows Prof. Moriarty took the left pathway. The note found for the left pathway is false. We can't say anything about the note found for the right pathway.

$$
\begin{array}{ll}
 & (pr \equiv \neg pl) \wedge (ir \equiv (ir \equiv \neg il)) \wedge (il \equiv pr) \\
= & \{ \quad \text{associativity and negation} \quad \} \\
 & (pr \equiv pl \equiv false) \wedge ((ir \equiv ir) \equiv il \equiv false) \wedge (il \equiv pr) \\
= & \{ \quad \text{reflexivity twice, substitution of equals for equals} \quad \} \\
 & (pr \equiv pl \equiv false) \wedge (il \equiv false) \wedge (il \equiv pr) \\
= & \{ \quad \text{substitution of equals for equals, reflexivity} \quad \} \\
 & (pl \equiv true) \wedge (pr \equiv false) \wedge (il \equiv false)
\end{array}
$$

Question 3: a) see fig. 0.1.

b) The sequence of Mex numbers for the pile of matches with substraction set $\{1, 4, 5\}$ is $0, 1, 0, 1, 2, 3, 2, 3, 0, 1, 0, 1, 2, 3$

c)

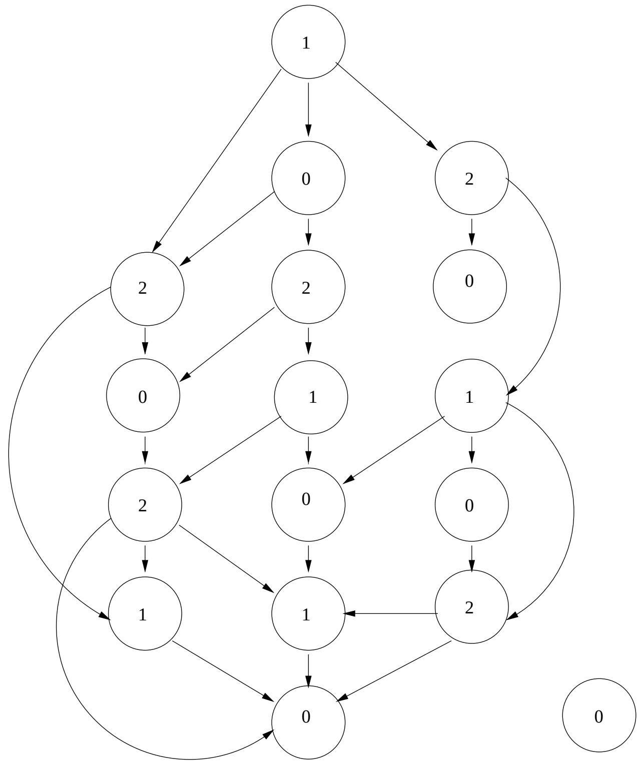

Figure 0.1 Question 3. Mex Numbers.

Left Game	Right Game	"losing" or winning move
G	7	5
E	36	losing
C	25	losing
I	19	1
M	20	losing

Table 0.2 Correct entries

Question 4:

a)

(i) $m + \frac{1}{2}n$ minutes: Start timing after lighting the m-minute fuse at one end. When it has burnt out, light the n-minute fuse at both ends.

(ii) $\frac{1}{2}(m+n)$ minutes. Start timing after lighting the m-minute fuse at both ends. When it has burnt out, light the n-minute fuse at both ends.

(iii) $m - \frac{1}{2}n$ minutes: Light the n-minute fuse at both ends and the m-minute fuse at one end. Start timing when the n-minute fuse has burnt out.

(iv) $\frac{1}{2}(m - \frac{1}{2}n)$ minutes: Light the n-minute fuse at both ends and the m-minute fuse at one end. When the n-minute fuse has burnt out, light the m-minute fuse at the other end, and start timing.

(v) $\frac{1}{2}n + \frac{1}{2}(m - \frac{1}{2}n)$ minutes: As for (iv) but start timing at the beginning.

b) See table 0.3. (This question was one a non-assessed coursework question. Of the sixteen, 12 are easy to construct.)

clocks in order	m,n := 28,32
$m+n$	60
$\frac{1}{2}m+n$	46
$\frac{1}{2}n+m$	44
n	32
$\frac{1}{2}(m+n)$	30
m	28
$\frac{1}{2}(n+\frac{1}{2}m)$	23
$\frac{1}{2}(m+\frac{1}{2}n)$	22
$n-\frac{1}{2}m$	18
$\frac{1}{2}n$	16
$\frac{1}{2}m$	14
$m-\frac{1}{2}n$	12
$\frac{1}{2}(n-\frac{1}{2}m)$	9
$\frac{1}{2}(m-\frac{1}{2}n)$	6
$n-m$	4
$\frac{1}{2}(n-m)$	2

Table 0.3 Maximising the Number of Fuse Clocks.

Question 5:

a) See the transition diagram in b). Four unreachable states that have been omitted are: the two states with four couples on one bank and the boat on the opposite bank, and the two states with the presidents and bodyguards on opposite banks and the boat on the same bank as the bodyguards.

b) See separate file: `4x3StateTransitionDiagram.pdf`.

c) There are $2{\times}4{\times}4$ different ways of getting the four couples across. This can be calculated using a topological search of the entire graph or, as suggested by the formula, by exploiting symmetry: the number of ways of getting from a leftmost state to a rightmost state (as displayed in the diagram) is 4 in each case that there is a path.

Question 6:

(This question is derived from the coursework.)

a)

The base case requires getting two people to cross the bridge.

The induction hypothesis requires that the optimal solution is known for $2n$ people.

Observe that to get $2(n+1)$ people to cross, $4n+1$ trips are needed, of which $2n$ are return trips. Consequently, at least 2 people never return. Clearly the two slowest should be among those who do not return.

The induction hypothesis is that it is possible to get $2n$ people across. The induction step involves getting the two slowest people across and returning the torch to the start side of tbe bridge. This requires choosing between two crossing strategies for the two slowest people.

The crossing strategies for the slowest two people are: 1) to get the two slowest cross together or 2) to get the slowest two people cross separately.

Choose the first strategy when $t_1 + t_{2n+1} \geq 2 \times t_2$.

Choose the second strategy when $t_1 + t_{2n+1} \leq 2 \times t_2$.

To complete the algorithm, apply the induction step on the remaining $2n$ people.

b)

The algorithm starts with the two slowest p_{16} and p_{20} . The strategy to choose would be to get them to cross together: $\langle p_1,p_5\rangle, [p_1], \langle p_{16},p_{20}\rangle, [p_5]$ (Angle brackets indicate a crossing in one direction, square brackets a crossing in the other direction.) After this sequence, just the two slowest have reached the opposite bank, and the torch is at the initial bank.

Now the current two slowest are p_6 and p_{14} . For these, the optimal strategy is to let them cross separately, with the fastest carrying the torch. This gives the sequence: $\langle p_1,p_6\rangle, [p_1], \langle p_1,p_{14}\rangle, [p_1]$.

The process is completed by p_1 and p_5 crossing together: $\langle p_1,p_5\rangle$.

The total time taken is $5{+}1{+}20{+}5{+}6{+}1{+}14{+}1{+}5$, i.e., 58 minutes.

c)

The two slowest people p_{25} and p_{30} should cross together, which gives $5{+}1{+}30{+}5$ or 41 minutes. From induction, the time taken to get the remaining six people are known. The total time taken will be just to add the time taken to get the 7th and 8th slowest people to cross with that of the time required to get the other six people to cross, i.e., $41 + 58$ or 99 minutes.