The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN SEMESTER 2011–2012

ALGORITHMIC PROBLEM SOLVING

Time allowed 2 hours

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

Answer FOUR questions.

Marks available for sections of questions are shown in brackets in the right-hand margin.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject-specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: Appendix with a summary of the mathematical laws discussed in the APS lectures and tutorials.

1 (a) Table 1.1 shows a list of expressions and assignments to the variables in the expressions. State for each, whether or not the expression is an invariant of the assignment. Justify your answer, either by using the invariant rule for assignment statements, or by giving a counterexample. Assume that m and n are integers. **(10)**

Assignment	Expression	Invariant?
$n := n{+}3$	$\mathsf{odd}(n)$	?
$m,n := m{+}6, n{+}4$	$2m < 3n$	?
$m,n := m{+}2, n{+}1$	$2m < 3n$	?
$m,n := n{+}3, m{+}1$	$\mathsf{even}(m) \equiv \mathsf{even}(n)$	?
$m,n := m{+}3, n{+}1$	$m \bmod 3$	?

Table 1.1 Fill in entries marked “?”

(b) A bag contains 3 blue balls and 4 red balls. Outside the bag, there is an unlimited supply of additional balls, each of which is blue or red.

The following process is repeated several times. One ball is removed from the bag. If the ball is blue, 4 blue balls and 2 red balls are returned to the bag. If the ball is red, 6 blue balls and 5 red balls are returned to the bag.

(i) Suppose b denotes the number of blue balls in the bag, and r denotes the number of red balls in the bag. Suppose the process of removing and returning balls is modelled as a nondeterministic simultaneous assignment to b and r. What is the assignment? **(5)**

(ii) Give an arithmetic expression depending on b and r that is an invariant of the assignment. What is the initial value of the expression? **(5)**

(iii) Table 1.2 shows different final values for the number of blue/red balls. Some are impossible. For each entry fill in the queries. If the state is impossible, write “impossible” in the entry. **(5)**

Number of blue balls	Number of red balls
?	25
?	10
?	33
8	?
18	?

Table 1.2 Fill in entries marked “?”

2 (a) Four children take part in a race. There are no dead-heats and each child finishes the race. (A dead-heat is when two children end in the same position.) After the race, three of the children make the following statements:

Ann: "Bob won"; "Cor was second"
Dee: "Bob was second"; "Cor was third"
Bob: "Cor was last"; "Ann was second"

We know that each child makes one and only one true statement.

When answering the following questions, use the notation cn to denote that the child whose name starts with the letter c ended the race in position n.

(i) Express formally the rule that there are no dead-heats. **(3)**

(ii) Express formally the rule that each child ends in at most one position. **(3)**

(iii) Express formally what is known from each of the statements. **(3)**

(iv) Deduce the final position of each of the children. Give full details of your calculation making clear which logical rules you use at each step. **(6)**

(b) Recall that a continued equivalence is an equivalence of a number of booleans that is evaluated associatively. For example,

$$\text{false} \equiv \text{true} \equiv \text{false}$$

is a continued equivalence of three booleans that evaluates to true. Suppose that there are n booleans in a continued equivalence and j of them are true. Let c be the value of the continued equivalence. (In the example above, n is 3, j is 1 and c is true.) This question is about proving that

$$c \equiv \text{even}(n) \equiv \text{even}(j)$$

is always true by induction on n.

(i) When n is zero, what are the values of j and c? **(2)**

(ii) Suppose that n is increased by the addition of one extra true value to the continued equivalence. Write an assignment that shows how the values of n, j and c change. **(2)**

(iii) Suppose that n is increased by the addition of one extra false value. Write an assignment that shows how the values of n, j and c change. **(2)**

(iv) Show that $c \equiv \text{even}(n) \equiv \text{even}(j)$ is an invariant of both assignments. State clearly any rule that you use in your calculation. **(4)**

3 Suppose a long strip of paper has been divided into squares. A single coin is placed on one of the squares. The objective is to displace the coin a given number of places to the right using a sequence of *moves*, which we now define. If there is at least one coin on any square, two extra coins can be added in the two adjacent squares. The reverse move is also possible: if there are three adjacent coins, two outer coins can be removed leaving just the middle coin. No other moves are possible, and there are no other restrictions on moves.

Fig. 3.1 illustrates the start and finish positions and the allowed moves.

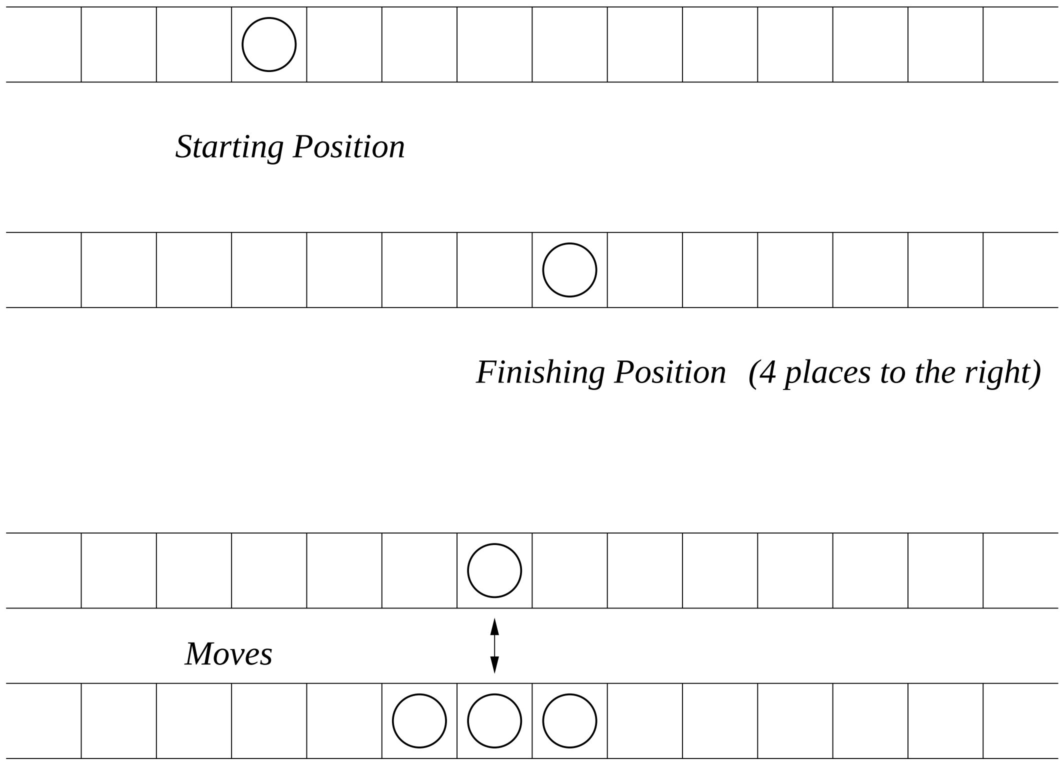

Figure 3.1 A one-dimensional game of solitaire

(a) Construct a sequence of moves that will displace a single coin four places to the right. (See part (b) for a hint that limits your search space.) Explain in words how the symmetry between left and right is exploited to decompose the problem in order to more easily find a solution. **(15)**

(b) Suppose you were to use brute-force search to solve part (a). Suppose you are told that a solution can be found using no more than 7 squares and using a maximum of 2 coins on each square. What is the maximum number of states you might have to examine in order to find a solution? Explain your answer. What is the general formula for the number of states you might have to examine if the solution can be found using no more than s squares and using a maximum of c coins on each square? **(5)**

(c) Discuss the difficulties of using brute-force search to solve a coin problem like this one. **(5)**

4 A group of n people wish to cross a bridge. It is dark, and it is necessary to use a torch when crossing the bridge, but they only have one torch between them. The bridge is narrow and only two people can be on it at any one time. Each person takes a different amount of time to cross the bridge; when two cross together they must proceed at the speed of the slowest. The people are all numbered from 1 to n , and person i takes t_i minutes to cross. The torch must be ferried back and forth across the bridge, so that it is always carried when the bridge is crossed.

Assume that the people all start at the left bank (so they wish to cross to the right bank). Below, we use the term *trip* to mean a crossing in either direction (from left to right or from right to left); a *forward* trip is a trip from left to right and a *return* trip is a trip from right to left. Assume that the minimal number of trips is used. That is, assume that there are two people in each forward trip and one person in each return trip.

(a) Suppose f and r count the number of forward and return trips, respectively. Suppose k counts the number of people on the right bank. Write an assignment statement that shows how k and f change when a forward trip is made; write an assignment statement that shows how k and r change when a return trip is made. **(4)**

(b) Show that $k+r-2\times f$ is an invariant of both assignment statements. (Give full details of the calculation showing explicitly how the assignment axiom is used.) What is its initial value? **(4)**

(c) What is the relation between f and r when the torch is on the right bank? What conclusion can you draw about the number of people that have not made a return trip when the torch is on the right bank? **(2)**

(d) When 3 people and the torch are at the right bank, how many people have made a return trip? Who would you choose to return in order to optimise the total travel time? What is the optimal value of the total travel time? Assume that the travel time of person i is $t.i$ and $t.1 \leq t.2 \leq t.3$. Prove that the value you give is indeed optimal. **(5)**

(e) Suppose we use (d) as the basis of an inductive algorithm to get $2n+3$ people across the bridge as fast as possible. For the inductive step, suppose there are $2(n+1)+3$ people and consider the two slowest people. What does (c) suggest? State two ways to get the two slowest people across the bridge and then return the torch to the left bank. Give a criterion for choosing between the two ways. (Assume that person i has travel time $t.i$ minutes and the people are numbered in increasing order of travel time.) **(5)**

(f) Suppose there are 7 people with crossing times 1, 4, 5, 6, 7, 8, 9. Use your solution to (e) to construct a sequence of trips that gets all across as fast as possible. Explain your solution. **(5)**

5 (a) Consider a game that is played with one pile of matches. Suppose a (strictly positive) number M is specified and a move in the game is to remove some number of matches that is at least 1 and at most M . The game is won by the player who removes the last match.

What is the mex number of a position where there are k matches remaining? **(3)**

(b) Suppose there are two piles of matches. A move is made by first choosing whether to move in the left or the right pile; the rule for moving in the left pile is to remove at least 1 and at most M matches, in the right pile the rule is to remove at least 1 and at most N matches (where M and N are fixed, strictly positive numbers).

Suppose there are i matches in the left pile and j matches in the right pile. What property of i and j guarantees that this is a winning position? If it is a winning position, how should the player move in order to place the opponent in a losing position? **(4)**

(c) Consider a matchstick game with one pile of matches. Suppose a move is to remove 1, 5 or 6 matches. Construct a table showing the mex number of each position such that the number of matches is at most 21 . **(8)**

(d) Consider a game that is the sum of two matchstick games. In the left game, 1, 5 or 6 matches may be removed; in the right game, at least 1 and at most 5 matches may be removed.

Complete the following table. If the position is a losing position, enter "losing"; if the position is a winning position enter a winning move in the form "L m" or "R m" where "L" indicates a move in the left game and "R" a move in the right game and m is the number of matches to be removed. **(10)**

Left Game	Right Game	"losing" or winning move
5	5	?
10	22	?
12	18	?
15	2	?
33	12	?

Table 5.1 Fill in entries marked "?"

6 As discussed in the lectures, a solution to the Tower of Hanoi problem is as follows.

$$H_0(d) = []$$
$$H_{n+1}(d) = H_n(\neg d) ; [\langle n{+}1 , d\rangle] ; H_n(\neg d)$$

($H_n(d)$ prescribes how to move the n smallest disks one-by-one from one pole to its neighbour in the direction d, following the rule of never placing a larger disk on top of a smaller disk.)

(a) Suppose $K(m,n)$ counts the number of times that disk m moves when the n smallest disks are moved. (Note that K is not parameterised by the direction d: you may assume that this number is independent of the direction in which the disks are moved.) Write down equations for $K(m,0)$, $K(n{+}1,n{+}1)$ and for $K(m,n{+}1)$ when $m \neq n{+}1$. What is the value of $K(m,n)$ when $m > n$? Use induction to show that $K(m,n) = 2^{n-m}$ when $1 \leq m \leq n$. **(6)**

(b) Suppose an additional restriction is added to how the disks may be moved. The restriction is that individual disks can only be moved clockwise. (For example, in order to move one disk in an anticlockwise direction, it is necessary to move it twice in a clockwise direction.)

A simple (but suboptimal) solution to this problem is based on the idea that moving n smallest disks from one pole to the next in an anticlockwise direction can always be achieved by moving the n disks twice in a clockwise direction.

Let $I_n(d)$ denote the sequence of moves used by this solution method to move the n smallest disks in direction d. Let aw abbreviate anticlockwise and cw abbreviate clockwise. Then the above idea is expressed by the equation

$$I_n(\text{aw}) = I_n(\text{cw}) ; I_n(\text{cw}) \quad .$$

Moving the n disks in a clockwise direction can then be done just as for the standard Tower of Hanoi problem:

$$I_0(\text{cw}) = []$$
$$I_{n+1}(\text{cw}) = I_n(\text{aw}) ; [n{+}1] ; I_n(\text{aw})$$

(The direction of movement of disk $n{+}1$ is omitted because individual moves are always clockwise.)

Suppose $K(m,n,d)$ counts the number of times that disk m moves when all n smallest disks are moved from one pole to the next in a clockwise direction. Note that K is parameterised by the direction d: more moves are needed to move the disks anticlockwise than clockwise.

Write an equation relating $K(m,n,\text{aw})$ and $K(m,n,\text{cw})$. **(2)**

Write equations for $K(m,0,\text{cw})$, $K(n{+}1,n{+}1,\text{cw})$ and for $K(m,n{+}1,\text{cw})$ when $m \neq n{+}1$. **(6)**

Use induction to determine a closed formulae for $K(m,n,cw)$ and $K(m,n,\text{aw})$. **(4)**

(c) Construct the sequence of moves defined by $I_3(\text{cw})$. Use your answer to explain why this is not an optimal solution to the problem. **(7)**

APPENDIX

Relevant Laws of Boolean Algebra

Equality of booleans is denoted by the symbol "$\equiv$" and inequality by "$\not\equiv$". Note that, in the symmetry, unit and negation laws, the associativity of boolean equality is assumed. That is, continued expressions are read associatively and not conjunctionally. The rules are typically used in combination with Leibniz's rule (also known as "substitution of equals for equals").

associativity: $[\ ((p \equiv q) \equiv r)\ \equiv\ (p \equiv (q \equiv r))\ ]$

symmetry: $[\ p \equiv q \equiv q \equiv p\ ]$

unit: $[\ \mathsf{true} \equiv p \equiv p\ ]$

negation: $[\ \neg p \equiv p \equiv \mathsf{false}\ ]$

inequivalence: $[\ (p \not\equiv q)\ \equiv\ \neg(p \equiv q)\ ]$

even: $[\mathsf{even}(m+n) \equiv \mathsf{even}(m) \equiv \mathsf{even}(n)$