The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN SEMESTER 2011–2012

ALGORITHMIC PROBLEM SOLVING

Time allowed 2 hours

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced.

Model Solutions

Note to external examiner. Two non-assessed courseworks have been set during the semester. All questions are based on the coursework questions. In most cases, the questions are essentially the same except with different parameters. The coursework questions are attached.

Question 1:

a)

Assignment	Expression	Invariant?
$n := n+3$	$\mathsf{odd}(n)$	no, $\mathsf{odd}(0) \neq \mathsf{odd}(3)$
$m, n := m+6, n+4$	$2m < 3n$	yes
$m, n := m+2, n+1$	$2m < 3n$	no, $2{\times}4 < 3{\times}3 \not\equiv 2{\times}6 < 3{\times}4$
$m, n := n+3, m+1$	$\mathsf{even}(m) \equiv \mathsf{even}(n)$	yes
$m, n := m+4, n+1$	$(m \bmod 3) - (n \bmod 3)$	yes

Table 0.1 Fill in entries marked "?"

2nd entry:

$$
\begin{array}{ll}
 & (2m < 3n)[m, n := m+6, n+4] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & 2(m+6) < 3(n+4) \\
= & \{ \quad \text{distributivity, arithmetic} \quad \} \\
 & 2m + 12 < 3n + 12 \\
= & \{ \quad \text{cancellation} \quad \} \\
 & 2m < 3n
\end{array}
$$

4th entry

$$
\begin{array}{ll}
 & (\mathsf{even}(m) \equiv \mathsf{even}(n))[m, n := n+3, m+1] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & \mathsf{even}(n+3) \equiv \mathsf{even}(m+1) \\
= & \{ \quad \text{distributivity} \quad \} \\
 & \mathsf{even}(n) \equiv \mathsf{even}(3) \equiv \mathsf{even}(m) \equiv \mathsf{even}(1) \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & \mathsf{even}(n) \equiv \mathsf{false} \equiv \mathsf{even}(m) \equiv \mathsf{false} \\
= & \{ \quad \text{properties of boolean equality} \quad \} \\
 & \mathsf{even}(m) \equiv \mathsf{even}(n) \quad .
\end{array}
$$

5th entry

$$
\begin{array}{ll}
 & (m \bmod 3)[m, n := m+3, n+1] \\
= & \{ \quad \text{definition of } m \bmod 3 \quad \} \\
 & (m - (m \div 3) \times 3)[m, n := m+3, n+1] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & (m+3) - ((m+3) \div 3) \times 3
\end{array}
$$

$=$ { $(m+3)\div 3 = m\div 3+1$ }

$(m+3)-(m+1)\times 3$

$=$ { distributivity, arithmetic }

$m-(m\div 3)\times 3$

$=$ { definition of $m \bmod 3$ }

$m \bmod 3$.

b)i) $b,r := b+3, r+2 \ \Box\ b,r := b+6, r+4$

ii) $2\times b - 3\times r$. Initially it is $2\times 3 - 3\times 4$, i.e. -6 .

iii)

Number of blue balls	Number of red balls
impossible	25
12	10
impossible	33
8	impossible
18	14

Table 0.2 Fill in entries marked "?"

Question 2:

(i) $[(pn \wedge pm = \text{false}) \vee (n=m)]$

(ii) $[(pn \wedge qn = \text{false}) \vee (p=q)]$

(iii)

Ann: $B1 \not\equiv C2$
Dee: $B2 \not\equiv C3$
Bob: $C4 \not\equiv A2$

(iv)

$(B1 \not\equiv C2) \wedge (B2 \not\equiv C3) \wedge (C4 \not\equiv A2)$

$=$ { distributivity }

$(B1 \wedge B2 \not\equiv B1 \wedge C3 \not\equiv C2 \wedge B2 \not\equiv C2 \wedge C3) \wedge (C4 \not\equiv A2)$

$=$ { no dead-heats, unique positions }

$(\text{false} \not\equiv B1 \wedge C3 \not\equiv \text{false} \not\equiv \text{false}) \wedge (C4 \not\equiv A2)$

$=$ { inequivalence }

$B1 \wedge C3 \wedge (C4 \not\equiv A2)$

$=$ { distributivity }

$B1 \wedge C3 \wedge C4 \not\equiv B1 \wedge C3 \wedge A2$

$=$ { unique positions }

$\text{false} \not\equiv B1 \wedge C3 \wedge A2$

$=$ { inequivalence }

$$B1 \wedge C3 \wedge A2$$

(b)

(i) $j = 0$ and $c = \text{true}$

(ii) $n, j \ := \ n+1, j+1$

(iii) $n, c \ := \ n+1, \neg c$

(iv) see below

1st assignment:

$$
\begin{array}{ll}
 & (c \equiv \text{even}(n) \equiv \text{even}(j))[n, j \ := \ n+1, j+1] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & c \equiv \text{even}(n+1) \equiv \text{even}(j+1) \\
= & \{ \quad \text{distributivity} \quad \} \\
 & c \equiv \text{even}(n) \equiv \text{false} \equiv \text{even}(j) \equiv \text{even}(1) \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & c \equiv \text{even}(n) \equiv \text{false} \equiv \text{even}(j) \equiv \text{false} \\
= & \{ \quad \text{properties of boolean equality} \quad \} \\
 & c \equiv \text{even}(n) \equiv \text{even}(j)
\end{array}
$$

2nd assignment:

$$
\begin{array}{ll}
 & (c \equiv \text{even}(n) \equiv \text{even}(j))[n, c \ := \ n+1, \neg c] \\
= & \{ \quad \text{assignment rule} \quad \} \\
 & \neg c \equiv \text{even}(n+1) \equiv \text{even}(j) \\
= & \{ \quad \text{negation, distributivity} \quad \} \\
 & c \equiv \text{false} \equiv \text{even}(n) \equiv \text{even}(1) \equiv \text{even}(j) \\
= & \{ \quad \text{arithmetic} \quad \} \\
 & c \equiv \text{false} \equiv \text{even}(n) \equiv \text{false} \equiv \text{even}(j) \\
= & \{ \quad \text{properties of boolean equality} \quad \} \\
 & c \equiv \text{even}(n) \equiv \text{even}(j)
\end{array}
$$

Question 3: (a) The solution is found by using symmetry between left and right. Starting from a coin in position 1 we determine how to get a coin in position 5:

```
  1
1 1 1
1 2 1 1
1 2 2 1 1
1 2 2 2 1 1
```

Similarly, starting from position 5 we determine how to get a coin in position 1:

```
  1 1 2 2 2 1
    1 1 2 2 1
```

```
      1 1 2 1
        1 1 1
          1
```

Now we have to get from the last state in the first diagram to the first state in the second diagram:

```
1 2 2 2 1 1
1 2 2 2 2 1 1
1 1 2 1 2 1 1
1 1 2 2 2 2 1
  1 1 2 2 2 1
```

Putting it altogether we get the following sequence:

```
  1
1 1 1
1 2 1 1
1 2 2 1 1
1 2 2 2 1 1
1 2 2 2 2 1 1
1 1 2 1 2 1 1
1 1 2 2 2 2 1
  1 1 2 2 2 1
    1 1 2 2 1
      1 1 2 1
        1 1 1
          1
```

(b) Each of the 7 squares can contain 0, 1 or 2 coins. There are thus 3^7 different states that may have to be searched. The general formula is c^s.

(c) The maximum number of squares and/or the maximum number of coins per square is not known in advance. This precludes the possibility of using (eg) depth-first search because there is no limit on the number of states that have to be searched. Also it is not known in advance whether or not the task can be achieved. It is much easier to construct examples where the task cannot be achieved than examples where it can be!

Question 4:

(a) $k,f \;:=\; k+2, f+1 \,;\; k,r \;:=\; k-1, r+1$

b)

$$(k+r-2\times f)[k,f \;:=\; k+2,f+1]$$

$$= \qquad \{ \quad \text{assignment rule} \quad \}$$

$$k+2+r-2\times(f+1)$$

$$= \qquad \{ \quad \text{distributivity, arithmetic} \quad \}$$

$$k+r-2\times f$$

$$(k+r-2\times f)[k,r \;:=\; k-1,r+1]$$

$=$ { assignment rule }

$k-1+r+1-2\times f$

$=$ { arithmetic }

$k+r-2\times f$

The initial value is 0.

(c) $f=r+1$. Since $k+r-2\times f$ is always 0, it follows that $k+r-2\times(r+1)=0$. That is, $r=k-2$. The number of return trips is 2 fewer than the number of people on the right bank.

(d) One person. Choose the fastest. The total time is $t.1 + t.1\uparrow t.2 + t.1\uparrow t.3$, i.e. $t.1+t.2+t.3$. If person 2 returns, the total time is $t.2+t.2+t.3$ and if person 3 returns it is $t.3+t.3+t.3$. Both of these are at least $t.1+t.2+t.3$.

(e) The two slowest should not return. Two ways they can get across are by travelling together or by travelling separately. In the first case, the time taken to get them across and return the torch to the left bank is at least $t.1 + 2\times t.2 + t.(2(n+1)+3)$. In the second case, the time taken is at least $2\times t.1 + t.(2(n+2)) + t.(2(n+1)+3)$. The choice between the two simplifies to $2\times t.2 \leq t.1+t.(2(n+2))$ versus $t.1+t.(2(n+2)) \leq 2\times t.2$.

(f) Applying the algorithm, it is better for the two slowest to cross together using the sequence: $+\{1,2\};-\{1\};+\{6,7\};-\{2\}$ since $2\times4<1+8$. For persons 4 and 5 it is better that they cross separately using the sequence $+\{1,4\};-\{1\};+\{1,5\};-\{1\}$ since $1+6<2\times4$. Finally, the three fastest cross using the sequence $+\{1,2\};-\{1\};+\{1,3\}$. The total time is $4\times1+3\times4+5+6+7+9$, i.e. 34.

Question 5: a) $k \bmod (M+1)$.

(b) $i \bmod (M+1) \neq j \bmod (M+1)$. The strategy is to equalise the modulo values. Choose the greater of the two and reduce the number of the matches in that pile by the difference. For example, if $i \bmod (M+1) > j \bmod (M+1)$, remove $i \bmod (M+1) - j \bmod (M+1)$ matches from the left pile. This is a number that is at least 1 and at most $i \bmod (M+1)$ (which is at most M).

(c) The repeating sequence of Mex numbers for the pile of matches with substraction set $\{1,5,6\}$ is

$$0,1,0,1,0,1,2,3,2,3,2 \quad .$$

(The solution makes the students check that the sequence repeats. If it doesn't they should be aware that there is an error.)

(d)

Left Game	Right Game	"losing" or winning move
5	5	R4
10	22	R2
12	18	L1
15	2	R2
33	12	losing

Table 0.3 Correct entries

Question 6:

(a)

$$\begin{array}{rcl} K(m,0) & = & 0 \\ K(n{+}1\,,n{+}1) & = & 1+2\times K(n{+}1\,,n) \\ K(m\,,n{+}1) & = & 2\times K(m\,,n) \end{array}$$

($K(n{+}1\,,n{+}1)=1$ is also acceptable.) When $m>n$, $K(m,n)=0$. It follows that

$$K(n{+}1\,,n{+}1)\ =\ 1\quad .$$

To show that, when $1\leq m\leq n$, $K(m,n)=2^{n-m}$, we use induction on $n{-}m$. The basis is

$$K(n{+}1\,,n{+}1)=1=2^{(n+1)-(n+1)}\quad .$$

For the induction step,

$$K(n{+}1\,,n{+}1{+}k{+}1)=2\times K(n{+}1\,,n{+}1{+}k)=2\times 2^k=2^{k+1}\quad .$$

(b)

$$\begin{array}{rcl} K(m,n,\text{aw}) & = & 2\times K(m,n,\text{cw}) \\ K(m,0,\text{cw}) & = & 0 \\ K(n{+}1\,,n{+}1\,,\text{cw}) & = & 1 \\ K(m\,,n{+}1\,,\text{cw}) & = & 2\times K(m,n,\text{aw}) \end{array}$$

Eliminating $K(m,n,\text{aw})$ we get the equations

$$K(n{+}1\,,n{+}1\,,\text{cw})=1$$

and

$$K(m\,,n{+}1\,,\text{cw})=4\times K(m,n,\text{cw})$$

when $m<n{+}1$. By induction, as above,

$$K(m,n,\text{cw})=4^{n-m}$$

and thus $K(m,n,\text{cw})=2\times 4^{n-m}$.

(c)

$I_3(\text{cw})$

$=\quad\{\quad \text{definition}\quad\}$

$I_2(\text{aw})\,;\,[3]\,;I_2(\text{aw})$

$=\quad\{\quad \text{definition}\quad\}$

$I_2(\text{cw})\,;I_2(\text{cw})\,;\,[3]\,;I_2(\text{cw})\,;I_2(\text{cw})$

$=\quad\{\quad \text{definition}\quad\}$

$I_1(\text{aw})\,;\,[2]\,;I_1(\text{aw})\,;I_1(\text{aw})\,;\,[2]\,;I_1(\text{aw})$

$;\quad [3]$

$$
\begin{array}{ll}
 & ;\quad I_1(\mathsf{aw})\,;\,[2]\,;\,I_1(\mathsf{aw})\,;\,I_1(\mathsf{aw})\,;\,[2]\,;\,I_1(\mathsf{aw}) \\
= & \qquad \{ \quad \text{definition} \quad \} \\
 & \quad I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,[2]\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,[2]\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw}) \\
 & ;\quad [3] \\
 & ;\quad I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,[2]\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw})\,;\,[2]\,;\,I_1(\mathsf{cw})\,;\,I_1(\mathsf{cw}) \\
= & \qquad \{ \quad \text{definition} \quad \} \\
 & \quad [1]\,;\,[1]\,;\,[2]\,;\,[1]\,;\,[1]\,;\,[1]\,;\,[1]\,;\,[2]\,;\,[1]\,;\,[1] \\
 & ;\quad [3] \\
 & ;\quad [1]\,;\,[1]\,;\,[2]\,;\,[1]\,;\,[1]\,;\,[1]\,;\,[1]\,;\,[2]\,;\,[1]\,;\,[1]
\end{array}
$$

The method is suboptimal because moving disk 1 four times has the same effect as moving it once. In general, $I_n(\mathsf{aw})\,;\,I_n(\mathsf{aw})$ has the same effect as $I_n(\mathsf{cw})$ but the former uses 4×4^n moves which is 4 times more than the latter. (—Note: The textbook gives the optimal solution.)