Equational Logic: Part 2

Roland Backhouse

March 6, 2001

Outline

We continue the axiomatisation of the propositional connectives.

The axioms for disjunction (the logical "or" of two statements) are added to the axioms for equivalence and negation, and then it is shown how to define the remaining logical connectives (conjunction, i.e. logical "and", "if" and "only if") in terms of these three primitives.

Axioms for Disjunction

Idempotence

$$p \vee p \ \equiv \ p \ .$$

Symmetry

$$p \vee q \ \equiv \ q \vee p \ .$$

Associativity

$$p \vee (q \vee r) \ \equiv \ (p \vee q) \vee r \ .$$

Simplifying Continued Disjunctions

The associativity of disjunction allows us to omit parentheses in continued disjunctions, as in for example

$$p \lor q \lor p \lor r \lor q \lor q \ .$$

The symmetry of disjunction means that the terms in such a continued disjunction can be rearranged at will, and the idempotence of disjunction means that multiple occurrences of the same term can be reduced to one. So the above expression would be simplified as follows:

$$p \lor q \lor p \lor r \lor q \lor q$$

$$= \qquad \{ \qquad \text{rearranging terms --- allowed because}$$

$$\text{disjunction is symmetric and associative} \quad \}$$

$$p \lor p \lor r \lor q \lor q \lor q$$

$$= \qquad \{ \qquad \text{idempotence of disjunction (applied three times)} \quad \}$$

$$p \lor r \lor q \ .$$

Axioms for Disjunction (Continued)

Distributivity

$$p \lor (q \equiv r) \;\; \equiv \;\; p \lor q \;\; \equiv \;\; p \lor r \;\; .$$

Excluded Middle

$$p \lor \neg p \;\; .$$

Using this basis we can derive, eg., false is a unit of disjunction:

$$p \lor \mathsf{false}$$

$$= \qquad \{ \qquad \text{definition of false} \quad \}$$

$$p \lor (\neg p \equiv p)$$

$$= \qquad \{ \qquad \text{disjunction distributes over equivalence} \quad \}$$

$$p \lor \neg p \;\equiv\; p \lor p$$

$$= \qquad \{ \qquad \text{excluded middle and idempotence of disjunction} \quad \}$$

$$\mathsf{true} \equiv p$$

$$= \qquad \{ \qquad \text{unit of equivalence} \quad \}$$

$$p \;\; .$$

Exercise 1   Prove that $p \lor \text{true} \equiv \text{true}$.

□

Conjunction

Golden Rule

$$p \wedge q \;\equiv\; p \;\equiv\; q \;\equiv\; p \vee q \;.$$

The golden rule can be seen as a definition of conjunction in terms of equivalence and disjunction if we read it as

$$(p \wedge q) \;=\; (p \;\equiv\; q \;\equiv\; p \vee q) \;.$$

But it can also be read in other ways. For example, the golden rule asserts the equality

$$(p \wedge q \;\equiv\; p) \;=\; (q \;\equiv\; p \vee q) \;.$$

This reading will be used later when we define logical implication. It can also be read as a definition of disjunction in terms of conjunction:

$$(p \wedge q \;\equiv\; p \;\equiv\; q) \;=\; (p \vee q) \;.$$

This reading is sometimes useful when in a calculation it is expedient to replace disjunctions by conjunctions.

Symmetry and Idempotence

$$p \wedge q$$

$$= \qquad \{ \qquad \text{golden rule} \quad \}$$

$$p \equiv q \equiv p \vee q$$

$$= \qquad \{ \qquad \text{equivalence and disjunction are symmetric} \quad \}$$

$$q \equiv p \equiv q \vee p$$

$$= \qquad \{ \qquad \text{golden rule, } p,q := q,p \quad \}$$

$$q \wedge p \quad .$$

Exercise 2 Prove that $p \wedge p \equiv p$ in the same way. (Begin with $p \wedge p$ and simplify.)

□

Absorption

$$p \vee (p \wedge q)$$

$=$ $\qquad \{ \qquad$ golden rule $\quad \}$

$$p \vee (p \equiv q \equiv p \vee q)$$

$=$ $\qquad \{ \qquad$ disjunction distributes over equivalence $\quad \}$

$$p \vee p \equiv p \vee q \equiv p \vee (p \vee q)$$

$=$ $\qquad \{ \qquad$ idempotence and associativity of disjunction $\quad \}$

$$p \equiv p \vee q \equiv p \vee q$$

$=$ $\qquad \{ \qquad$ unit/reflexivity law of equivalence $\quad \}$

$$p \quad .$$

Associativity

Now we prove that conjunction is associative. First, we observe a lemma.

$$p \wedge (q \wedge r)$$

$$= \qquad \{ \qquad \text{golden rule: } p,q := p, q \wedge r \ \}$$

$$p \equiv q \wedge r \equiv p \vee (q \wedge r)$$

$$= \qquad \{ \qquad \text{golden rule: } p,q := q,r \ \}$$

$$p \equiv q \equiv r \equiv q \vee r \equiv p \vee (q \equiv r \equiv q \vee r)$$

$$= \qquad \{ \qquad \text{distributivity of disjunction over equivalence} \ \}$$

$$p \equiv q \equiv r \equiv q \vee r \equiv p \vee q \equiv p \vee r \equiv p \vee q \vee r \ .$$

Associativity (Continued)

To complete our task we use the lemma twice, once with a different
permutation of the variables:

$$p \land (q \land r)$$

$$= \qquad \{ \qquad \text{above} \quad \}$$

$$p \equiv q \equiv r \equiv q \lor r \equiv p \lor q \equiv p \lor r \equiv p \lor q \lor r$$

$$= \qquad \{ \qquad \text{equivalence is symmetric, disjunction is symmetric} \quad \}$$

$$r \equiv p \equiv q \equiv p \lor q \equiv r \lor p \equiv r \lor q \equiv r \lor p \lor q$$

$$= \qquad \{ \qquad \text{above, } p,q,r := r,p,q \quad \}$$

$$r \land (p \land q)$$

$$= \qquad \{ \qquad \text{conjunction is symmetric} \quad \}$$

$$(p \land q) \land r \quad .$$

Distributivity of disjunction over conjunction

$$(p \vee q) \wedge (p \vee r)$$

$= \qquad \{ \qquad \text{golden rule, } p,q := p \vee q, p \vee r \quad \}$

$$p \vee q \equiv p \vee r \equiv p \vee q \vee p \vee r$$

$= \qquad \{ \qquad \text{continued disjunctions} \quad \}$

$$p \vee q \equiv p \vee r \equiv p \vee q \vee r$$

$= \qquad \{ \qquad \text{distributivity of disjunction over equivalence} \quad \}$

$$p \vee (q \equiv r \equiv q \vee r)$$

$= \qquad \{ \qquad \text{golden rule, } p,q := q,r \quad \}$

$$p \vee (q \wedge r) \quad .$$

Distributivity of conjunction over disjunction

$$(p \wedge q) \vee (p \wedge r)$$

$=$ { distributivity of disjunction over conjunction

(just proved) }

$$((p \wedge q) \vee p) \wedge ((p \wedge q) \vee r)$$

$=$ { absorption (see above) }

$$p \wedge ((p \wedge q) \vee r)$$

$=$ { distributivity of disjunction over conjunction

(symmetric version) }

$$p \wedge (p \vee r) \wedge (q \vee r)$$

$=$ { absorption }

$$p \wedge (q \vee r) \ .$$

Implication

Definition of "if"

$$p \Leftarrow q \;\equiv\; p \;\equiv\; p \vee q \;.$$

$$p \Leftarrow q \;\equiv\; q \;\equiv\; p \wedge q \;.$$

Definition of "only if"

$$p \Rightarrow q \;\equiv\; q \;\equiv\; p \vee q \;.$$

$$p \Rightarrow q \;\equiv\; p \;\equiv\; p \wedge q \;.$$

Exercise 3   Prove that $p \Leftarrow \mathsf{true} \equiv p$ and $p \Leftarrow \mathsf{false} \equiv \mathsf{true}$.

□

Modus Ponens

The advantage of using equations over other methods for defining the logical connectives is the opportunity to substitute equals for equals. The definition of $p \Leftarrow q$ provides good examples of this.

$$(p \Leftarrow q) \wedge q$$

$$= \qquad \{ \qquad \text{true is unit of equivalence} \quad \}$$

$$(p \Leftarrow q) \wedge (q \equiv \text{true})$$

$$= \qquad \{ \qquad \text{substitution of equals for equals: the value}$$

$$\text{true is substituted for } q \text{ in the term } p \Leftarrow q \quad \}$$

$$(p \Leftarrow \text{true}) \wedge (q \equiv \text{true})$$

$$= \qquad \{ \qquad p \Leftarrow \text{true} \equiv p \quad \}$$

$$p \wedge q \quad .$$

$$(p \Leftarrow q) \land (q \Leftarrow r)$$

=   {   definition   }

$$(p \equiv p \lor q) \land (q \equiv q \lor r)$$

=   {   conjunction is idempotent and symmetric   }

$$(p \equiv p \lor q) \land (q \equiv q \lor r) \land (p \equiv p \lor q)$$

=   {   substitution of equals for equals:

         the rightmost occurrence of q is replaced by $q \lor r$ }

$$(p \equiv p \lor q) \land (q \equiv q \lor r) \land (p \equiv p \lor q \lor r)$$

=   {   substitution of equals for equals:

         the rightmost occurrence of $p \lor q$ is replaced by p }

$$(p \equiv p \lor q) \land (q \equiv q \lor r) \land (p \equiv p \lor r)$$

=   {   definition   }

$$(p \Leftarrow q) \land (q \Leftarrow r) \land (p \Leftarrow r)$$

$\Rightarrow$ {   weakening   }

$$p \Leftarrow r \quad .$$