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On difunctions

Roland Backhouse [a], José Nuno Oliveira [b,*]

[a] School of Computer Science, University of Nottingham, Nottingham NG8 1BB, UK
[b] Departamento de Informática, Universidade do Minho, Gualtar, Braga, Portugal

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ABSTRACT

The notion of a difunction was introduced by Jacques Riguet in 1948. Since then it has played a prominent role in database theory, type theory, program specification and process theory. The theory of difunctions is, however, less known in computing than it perhaps should be. The main purpose of the current paper is to give an account of difunction theory in relation algebra, with the aim of making the topic more mainstream.

As is common with many important concepts, there are several different but equivalent characterisations of difunctionality, each with its own strength and practical significance. This paper compares different proofs of the equivalence of the characterisations.

A well-known property is that a difunction is a set of completely disjoint rectangles. This property suggests the introduction of the (general) notion of the "core" of a relation; we use this notion to give a novel and, we believe, illuminating characterisation of difunctionality as a bijection between the classes of certain partial equivalence relations.

"Now - now, as low I stooped, thought I,
I will see what this snowdrop is;
So shall I put much argument by,
And solve a lifetime's mysteries."
Walter John de la Mare (1873-1956), 1929

1. Introduction

This paper is a tribute to our colleague and friend Luís Soares Barbosa, on the occasion of his 60th birthday. Well-versed in the poetic and mathematical discourses, Luís values concise expression, be it in the aesthetic or formal side of his interests: *"beauty is our business"*, in the words of Dijkstra [9]. His doctoral research [3] relies on the *coalgebra* concept [19] to achieve generality and economy of expression, in particular when combined with the relational algebraic approach [7,6].

Binary relations play a major role in the coalgebraic theory of concurrent processes, which is one of Luís main research interests. Coalgebras are functions, and as such particular cases of relations; bisimulations are relations, possibly equivalences, as is the case of bisimilarity; and so on and so forth — the list is long.

A concept that unifies many of these situations is that of a *difunction*. The notion of a difunction was introduced by Riguet in 1948 [16]. Since then it has played a prominent role in database theory [14]; it also has a significant role, as yet not fully appreciated, in the development of a theory of types that includes subtyping and types with laws [22].

* Corresponding author.
E-mail addresses: roland.backhouse@nottingham.ac.uk (R. Backhouse), jno@di.uminho.pt (J.N. Oliveira).

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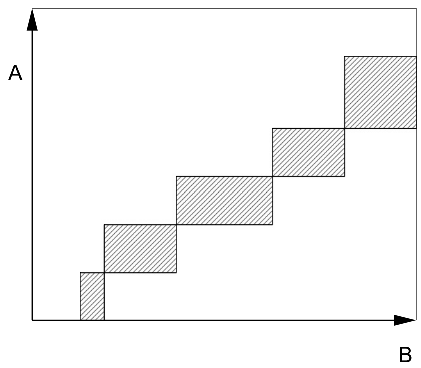

Fig. 1. Mental picture of a difunctional relation.

Difunctions are also used in program specification [13,15] and in the characterisation of certain types of simulations in process theory [12]. The theory of difunctions is, however, far less known in computing than it should be. The main purpose of the current paper is to give an account of difunction theory in relation algebra, with the aim of making the topic more mainstream.

As is common with many important concepts, there are several different but equivalent characterisations of difunctionality, each with its own strength and practical significance. The primary contribution of this paper is to compare different proofs of the equivalence of the characterisations.

Fig. 1 is a "mental picture" of a difunctional relation of type $A{\sim}B$. Informally (as first formulated by Riguet [16]), a difunction is a (heterogeneous) relation that comprises a collection of "completely disjoint rectangles[1]".

A special case of a difunction is a partial equivalence relation — "per" for short. Whereas a difunction is pictured as a collection of completely disjoint *rectangles*, a per is pictured as a collection of disjoint[2] *squares*, each square of which pictures an equivalence class.

Like difunctions, there are several different but equivalent characterisations of pers. In the same way that the characterisation of pers as collections of disjoint squares generalises to difunctions as collections of completely disjoint rectangles, each of the different characterisations of pers generalises to a different characterisation of difunctions. It is well-known, for example, that a (partial) equivalence relation can be characterised by a (single) functional relation that maps each element in the domain of the relation to the equivalence class to which it belongs. Similarly —as first formulated by Riguet [16]— a difunction can be characterised as a *pair* of functional relations that maps each pair of related elements to the rectangle to which they belong. The simplest characterisation of a per is, of course, a relation that is symmetric and transitive. (An equivalence relation is also reflexive.) Symmetry and transitivity can be combined, very concisely, into a single equation; similarly, the definition of a difunction can be given, also very concisely, as a single equation: see Theorem 25 (in particular 25(ii)).

In this paper, we assume that the reader has a good understanding of the different characterisations of (partial) equivalence relations outlined above in order to structure our account of the characterisations of difunctions.

Paper structure and summary of contributions To set the scene, the primary contribution of this paper begins in section 3 with a formalisation of a bag/set of "completely disjoint rectangles"; the main result is Theorem 23 which characterises such a bag/set in terms of a pair of functional relations.

Section 4 is where we prove that —as the name "di"function suggests— a difunctional relation is characterised by a pair of functional relations. We give two quite different constructions in subsection 4.3, which are shown to be isomorphic.

The steps taken in the generalisation of properties of pers to properties of difunctions can be taken yet further: we introduce the (general) notion of the "core" of an arbitrary relation; in section 5, we use this notion to give a novel and, we believe, illuminating characterisation of difunctionality as a bijection between the classes of certain partial equivalence relations.

Before we can begin on this endeavour, it is necessary to summarise the axiom system that we call "relation algebra" (section 2). Central to our relational reasoning are the left and right "domains" of a relation, of which there are two kinds: *coreflexive* domains and *per* domains, both playing a vital role in what follows. Section 6 discusses the relative merits of the proofs we have presented, in particular with respect to our limited use of points and entirely point-free calculations. We also briefly mention ongoing work on "cores" and "indexes" of a relation.

[1] See Definition 19 for a formal definition of "completely disjoint rectangles".

[2] For rectangles, "completely disjoint" is a special case of "disjoint"; for squares there is no distinction.

2. Elements of relation algebra

The calculations that follow make use of an axiomatic formulation of relation algebra. This section gives a brief summary of this algebra. Readers less fluent in "relation thinking" are referred to Appendix A for details. For a more comprehensive account of the several layers of the axiomatic formulation see reference [4].

Basics A (heterogeneous) relation of type $A{\sim}B$ is a subset of the cartesian product $A{\times}B$, i.e. an element of the powerset lattice $2^{A\times B}$. Symbol $\top\!\!\top$ (resp. $\perp\!\!\perp$) denotes the top (resp. bottom) element of the lattice. Given relations R and S of types $A{\sim}B$ and $B{\sim}C$, respectively, the *composition* of R and S is a relation of type $A{\sim}C$ denoted by $R{\circ}S$. Composition is associative. The *identity* relation on type A is denoted by I_A, often abbreviated to I when types are clear from the context. I and $\perp\!\!\perp$ are the unit and zero of composition, respectively.

If R is a relation of type $A{\sim}B$, the *converse* of R, denoted by $R^{\cup}$ (pronounced R "wok") is of type $B{\sim}A$. Converse is a poset isomorphism meaning that, in particular, $\perp\!\!\perp^{\cup}{=}\perp\!\!\perp$, $\top\!\!\top^{\cup}{=}\top\!\!\top$, $I^{\cup}{=}I$ and $(R{\circ}S)^{\cup} = S^{\cup}\circ R^{\cup}$, for suitably typed R and S.

Points and pairs A *coreflexive of type* A is a relation p such that $p\subseteq I_A$. Our axiom system uses coreflexives of a given type to represent sets of elements of that type. If A is a type, we use a, a' etc. to denote "points" of type A. Points are "singleton" coreflexives and, as such, enjoy a number of useful properties (see Appendix A).

In general, given points a and b of types A and B, respectively, the relation $a{\circ}\top\!\!\top{\circ}b$ (of type $A\sim B$) represents the pair (a,b). Thus $a{\circ}\top\!\!\top{\circ}b \subseteq R$ means that a and b are related by R. The *saturation* property states that a relation R is the set of its pairs:

$$\langle\forall R\ ::\ R = \langle\cup a{,}b : a{\circ}\top\!\!\top{\circ}b \subseteq R : a{\circ}\top\!\!\top{\circ}b\rangle\rangle\ .$$

All pairs of the *identity relation* are made of the same element: $a{\circ}\top\!\!\top{\circ}a' \subseteq I_A \ \equiv\ a{=}a'$.

Relations of the form $R{\circ}b{\circ}S$, where b is a point, play a central role in what follows. The interpretation of $R{\circ}b{\circ}S$ is a relation that holds between points a and c iff R holds between a and b, and S holds between b and c:

$$a{\circ}\top\!\!\top{\circ}c \subseteq R{\circ}b{\circ}S \ \ \equiv\ \ a{\circ}\top\!\!\top{\circ}b \subseteq R\ \wedge\ b{\circ}\top\!\!\top{\circ}c \subseteq S\ . \tag{1}$$

An alternative way of representing sets in relation algebra is to use "squares". A relation R is a *square* iff $R = R\circ\top\!\!\top\circ R^{\cup}$. Points are squares.

Factors Given R and S of types $A{\sim}B$ and $A{\sim}C$, respectively, the *right factor* $R\backslash S$ (of S) of type $B{\sim}C$ is defined by the Galois connection, for all T (of type $B{\sim}C$),

$$T \subseteq R\backslash S\ \equiv\ R{\circ}T \subseteq S\ . \tag{2}$$

Dually, the *left factor* (of R), denoted by R/S, is given by the Galois connection,

$$T \subseteq R/S\ \equiv\ T{\circ}S \subseteq R\ , \tag{3}$$

for R, S and R/S of types $A{\sim}B$, $C{\sim}B$ and $A{\sim}C$, respectively. As adjoints of Galois connections, *factors* enjoy a rich set of useful properties — see Appendix A.

Domain operators Given relation R of type $A{\sim}B$, the *left domain* $R{<}$ of R is a relation of type A defined by the equation

$$R{<}\ =\ I_A \cap R\circ R^{\cup} \tag{4}$$

and the *right domain* $R{>}$ of $R$is a relation of type B is defined by the equation

$$R{>}\ =\ I_B \cap R^{\cup}\circ R\ . \tag{5}$$

The name "domain operator" is chosen because of properties such as, for instance,

$$R{<}\circ R\ =\ R\ =\ R\circ R{>}\ . \tag{6}$$

In words, $R{<}$ and $R{>}$ represent the set of points on the left and on the right on which the relation R is "defined", i.e. its left and right "domains".

The domain operators play a dominant role in relation algebra. As lower adjoints in Galois connections, they are monotonic (with respect to the subset relation) and distribute through the union of relations. They also help in expressing the meaning of *factors* in relation algebra. (Details in Appendix A.)

Functionality A relation R of type $A{\sim}B$ is said to be *left-functional* iff $R\circ R^{\cup} = R{<}$. Equivalently, R is *left-functional* iff $R\circ R^{\cup} \subseteq I_A$. It is said to be *right-functional* iff $R^{\cup}\circ R = R{>}$ (equivalently, $R^{\cup}\circ R \subseteq I_B$). A relation R is said to be a *bijection* iff it is both left- and right-functional.

Rather than left- and right-functional, the more common terminology is "functional" and "injective" but publications differ on which of left- or right-functional is "functional" or "injective". We choose to abbreviate "left-functional" to *functional* and to use the term *injective* instead of right-functional. Other authors make the opposite choice.

The properties of functional relations stem from the observation that functionality can be defined via a Galois connection. Specifically, the relation f is functional iff, for all relations R and S (of appropriate type),

$$f\circ R \subseteq S \quad\equiv\quad f{>}\circ R \subseteq f^{\cup}\circ S \;\;, \tag{7}$$

which is equivalent to the property $f\circ f^{\cup} \subseteq I$.[3] Again by converse-duality, one has:

$$R\circ g^{\cup} \subseteq S \quad\equiv\quad R\circ g{>} \subseteq S\circ g \;\;. \tag{8}$$

Several theorems we present "characterise" classes of relations in terms of functional relations. Typically these characterisations are unique "up to isomorphism".

Power transpose Given relations R of type $A{\sim}B$ and S of type $A{\sim}C$, the symmetric *right-division* is a relation of type $B{\sim}C$ defined in terms of *right* factors as $R\backslash\!\!\backslash S$ where

$$R\backslash\!\!\backslash S \;\;=\;\; R\backslash S \cap (S\backslash R)^{\cup} \;\;. \tag{9}$$

Dually, given relations R of type $B{\sim}A$ and S of type $C{\sim}A$, the symmetric *left-division* is a relation of type $B{\sim}C$ defined in terms of *left* factors as $R/\!\!/S$ where

$$R/\!\!/S \;\;=\;\; R/S \cap (S/R)^{\cup} \;\;. \tag{10}$$

Given a relation R of type $A{\sim}B$, the *(left) power transpose* of R is a total function, denoted in this paper by ΓR, of type $B\rightarrow 2^A$. A pointwise definition of the (left) power transpose (using traditional set notation) is $\Gamma R.b \;=\; \{a\,|\,a\;R\;b\}$. A property of ΓR (a total function) that we shall use is, for all R and S,

$$(\Gamma R)^{\cup}\circ\Gamma S \;\;=\;\; R\backslash S \cap (S\backslash R)^{\cup} \;\;, \tag{11}$$

that is:

$$(\Gamma R)^{\cup}\circ\Gamma S \;\;=\;\; R\backslash\!\!\backslash S \;\;. \tag{12}$$

Pers and per domains A relation R is a *partial equivalence relation* (abbreviated to "*per*") iff it is symmetric and transitive, that is, $R = R^{\cup}$ and $R\circ R \subseteq R$ hold. An equivalence relation is a *reflexive per*. Reflexivity means that the left domain, the right domain, the source and the target of the relation are all the same.

The relation $R\backslash\!\!\backslash R$ is an equivalence relation, whose cancellation property

$$R\circ R\backslash\!\!\backslash R \;=\; R \tag{13}$$

will be useful in the sequel. Relations $R{>}\circ R\backslash\!\!\backslash R$ and $R/\!\!/R\circ R{<}$ are pers. Following [22], who advocates that pers are better than coreflexives to capture relation "domains", we make use of the *right per-domain* of R, denoted $R{\succ}$, and the *left per-domain* of R, denoted $R{\prec}$, defined by:

$$R{\succ} \;=\; R{>}\circ R\backslash\!\!\backslash R \tag{14}$$

$$R{\prec} \;=\; R/\!\!/R\circ R{<} \tag{15}$$

The left and right per-domains are called "domains" because, like the coreflexive domains, they enjoy the properties[4]:

$$R{\prec}\circ R \;=\; R \;=\; R\circ R{\succ} \;\;. \tag{16}$$

Appendix A gives a number of properties of the two per-domains that are very useful for what is to come in this paper — see Lemmas A.18, A.23 and A.24, for instance.

[3] Although the equivalence doesn't immediately fit the standard definition of a Galois connection, it can be turned into standard form by restricting the range of the dummy R to relations that satisfy $f{>}\circ R = R$, i.e. relations R such that $R{<} \subseteq f{>}$.

[4] The combination of the two per-domains enables the definition of what we call the "core" of a relation (see section 5).

3. Collections of rectangles and squares

Now that we have completed our review of relation algebra, we may embark on our endeavour to fully comprehend Riguet's notion of a difunction. We begin with the mental picture of a collection of completely disjoint rectangles illustrated by Fig. 1.

As is well-known, an equivalence relation *partitions* its domain into a set of disjoint *classes*. Also well-known is that the existence of such a partitioning is precisely formulated by the function that maps an element of the domain to its *equivalence class*: two elements are equivalent if and only if their equivalence classes are equal. When represented by relations, equivalence classes are squares. The theory of difunctions generalises this partitioning property to "completely disjoint" rectangles. This section lays the foundations for this theory. Specifically, Theorem 23 formulates a correspondence between *pairs* of functional relations and sets of completely disjoint rectangles. Section 3.1 introduces basic definitions and properties.

3.1. Completely disjoint rectangles

A *rectangle* is a (heterogeneous) relation R such that $R = R\circ\top\!\!\top\circ R$. An example of a rectangle is the "pair" $a\circ\top\!\!\top\circ b$ where a and b are points. More generally, we have:

Lemma 17. *For all relations R and S, $R\circ\top\!\!\top\circ S$ is a rectangle. It follows that $R\circ T\circ S$ is a rectangle if T is a rectangle. In particular, if R has type $A{\sim}B$, S has type $B{\sim}C$, and b is a point of type B, the relation $R\circ b\circ S$ is a rectangle.* $\square$

An example of Lemma 17 is that a square is a rectangle: recall that a relation R is a *square* iff $R = R\circ\top\!\!\top\circ R^{\cup}$. Properties of squares are typically obtained by specialising properties of rectangles.

Definition 18 *(Indexed bag/set).* Suppose $\mathcal{R}$ is a function with source K. Then $\mathcal{R}$ is said to be a *bag indexed by* K. The values $\mathcal{R}.k$, where k ranges over K, are said to be the *elements* of $\mathcal{R}$. In the case that $\mathcal{R}$ is injective, it is said to be an *indexed set*. $\square$

The distinction between "bag" and "set" in Definition 18 emphasises the fact that the same element may occur repeatedly in an indexed bag whereas each element occurs exactly once in an indexed set. That is, an indexed set $\mathcal{R}$ has the property that, for all j and k in K,

$$\mathcal{R}.j = \mathcal{R}.k \;\equiv\; j = k \;.$$

We normally apply Definition 18 to bags/sets of rectangles. Specifically, suppose A, B and K are types and $\mathcal{R}$ is a function with source K and target rectangles of type $A{\sim}B$. Then $\mathcal{R}$ is said to be an *indexed bag of rectangles*; it is an indexed *set* of rectangles if it is injective.

Two relations R and S are *disjoint* if $R\cap S = \perp\!\!\perp$. One can show that, for all rectangles R and S,

$$R\cap S = \perp\!\!\perp \;\equiv\; R{<}\cap S{<} = \perp\!\!\perp \,\vee\, R{>}\cap S{>} = \perp\!\!\perp \;.$$

The definition of "completely" disjoint strengthens the disjunction to a conjunction.

Definition 19 *(Completely disjoint).* Two rectangles R and S are said to be *completely disjoint* iff

$$R{<}\cap S{<} = \perp\!\!\perp \;\wedge\; R{>}\cap S{>} = \perp\!\!\perp \;.$$

Suppose $\mathcal{R}$ is an indexed bag of rectangles. Then $\mathcal{R}$ is said to be a *bag of completely disjoint rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$\mathcal{R}.j \neq \mathcal{R}.k \;\equiv\; (\mathcal{R}.j){<}\cap(\mathcal{R}.k){<} = \perp\!\!\perp \wedge (\mathcal{R}.j){>}\cap(\mathcal{R}.k){>} = \perp\!\!\perp \;.$$

$\mathcal{R}$ is said to be a *set* of completely disjoint rectangles iff in addition it is injective. That is, $\mathcal{R}$ is a *set of completely disjoint rectangles* iff, for all j and k in the index set of $\mathcal{R}$,

$$j \neq k \;\equiv\; (\mathcal{R}.j){<}\cap(\mathcal{R}.k){<} = \perp\!\!\perp \wedge (\mathcal{R}.j){>}\cap(\mathcal{R}.k){>} = \perp\!\!\perp \;. \quad \square$$

When constructing a bag/set of rectangles, the verification that the bag/set is completely disjoint is achieved by mutual implication. The "if" part is established by proving its contrapositive. That is, the proof obligation becomes to show that, for all indices j and k,

$$\mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j){<}\cap(\mathcal{R}.k){<} \neq \perp\!\!\perp \wedge (\mathcal{R}.j){>}\cap(\mathcal{R}.k){>} \neq \perp\!\!\perp$$

which simplifies to, for all j,

$$\mathcal{R}.j \neq \perp\!\!\!\perp \ .$$

(The same simplification is valid whether the construction yields a bag or a set.) Thus the first step is to show that the construction yields non-empty elements. The "only-if" part is to show that, for all indices j and k,

$$\mathcal{R}.j \neq \mathcal{R}.k \ \Rightarrow\ (\mathcal{R}.j)_< \cap (\mathcal{R}.k)_< = \perp\!\!\!\perp \ \wedge\ (\mathcal{R}.j)_> \cap (\mathcal{R}.k)_> = \perp\!\!\!\perp \ .$$

For this part, the following lemma is exploited.

Lemma 20. *For all relations R and S,*

$$R_< \cap S_< = \perp\!\!\!\perp \quad \equiv \quad R^\cup \circ S = \perp\!\!\!\perp \ .$$

Symmetrically,

$$R_> \cap S_> = \perp\!\!\!\perp \quad \equiv \quad R \circ S^\cup = \perp\!\!\!\perp \ .$$

(Proof in Appendix B.) □

Here is an example of such a construction.

Lemma 21. *Suppose f and g are relations with common target C such that*

$$f \circ f^\cup \ = \ f_< \ = \ g \circ g^\cup \ = \ g_< \ .$$

Then the relation $f^\cup \circ g$ is the union of an indexed set of completely disjoint rectangles. Specifically, with dummy c ranging over points of type C,

$$f^\cup \circ g \ = \ \langle \cup c : c \subseteq g_< : f^\cup \circ c \circ g \rangle \ .$$

(Proof in Appendix B.) □

We now establish the converse of Lemma 21.

Lemma 22. *Suppose relation R is the union of a set of completely disjoint rectangles. Then*

$$\langle \exists f,g \ :\ f \circ f^\cup \ = \ f_< \ = \ g \circ g^\cup \ = \ g_< \ :\ R = f^\cup \circ g \rangle \ .$$

(Proof in Appendix B.) □

Theorem 23. *A relation R is the union of a set of completely disjoint rectangles if and only if*

$$\langle \exists f,g \ :\ f \circ f^\cup \ = \ f_< \ = \ g \circ g^\cup \ = \ g_< \ :\ R = f^\cup \circ g \rangle \ .$$

Proof. "If" is Lemma 21 and "only-if" is Lemma 22. □

4. Difunctions

As Riguet remarked, difunctional relations generalise both functional relations [16] and pers [17, "quasi-equivalences"] in the sense that a difunction is characterised by a pair of functional relations whilst a per is characterised by a single functional relation; equivalently, a difunction is a union of completely disjoint rectangles whilst a per is the union of disjoint squares.[5] We present different calculational proofs of Theorem 26 in section 4.3 using both point-free and pointwise calculations, with a view to gaining insight into the efficacy and aesthetics of the calculational method. Note that, although the proofs are quite different, the constructed characterisations are essentially the same, as is made precise in section 4.2. Theorem 27 is a straightforward combination of Theorem 26 and the (already-proven) Theorem 23.

[5] See Theorems 26 and 27 below.

4.1. Formal definition and characterisation

In this subsection we give the formal definition of a "difunctional relation" and state the theorem (Theorem 26) that we prove in subsection 4.3. Theorem 26 uses the notion of a "characterisation" of a difunctional relation; this notion is also introduced in this subsection.

Formally, relation R is *difunctional* equivales

$$R \circ R^{\cup} \circ R \subseteq R \ . \tag{24}$$

As for pers, there are several equivalent definitions of "difunctional" —see Lemma A.24 in the appendix. We begin with the simplest:

Theorem 25. *For all R, the following statements are all equivalent.*

(i) *R is difunctional (i.e. $R \circ R^{\cup} \circ R \subseteq R$),*
(ii) $R = R \circ R^{\cup} \circ R$,
(iii) $R{\succ} = R^{\cup} \circ R$,
(iv) $R{\prec} = R \circ R^{\cup}$,
(v) $R = R \cap (R \backslash R / R)^{\cup}$. $\square$

The equivalence of 25(i) and 25(ii) is well-known and due to Riguet [16]; the equivalence of 25(i), (iii) and (iv) is due to Voermans [22]. Definition (24) is the most useful when it is required to establish that a particular relation is difunctional, whereas Properties 25(ii)-(iv) are more useful when it is required to exploit the fact that a particular relation is difunctional.

The right side of 25(v) is what Riguet [18] calls the différence of relation R. (Thus, the theorem states that a relation is a difunction iff it equals its différence.) Riguet's terminology is based on a different definition involving nested complements. For reasons explained in detail elsewhere [2], we prefer to call it the *diagonal* of R. See also [2] for a detailed analysis of its properties.[6]

Theorem 25 gives several *point-free* definitions of the notion of difunctionality all of which are equivalent but none of which in any way resembles the mental picture shown in Fig. 1 of a collection of completely disjoint rectangles. The key to linking the formal definition to the mental picture is the combination of Theorem 23 and the following theorem.[7]

Theorem 26. *For all relations R,*

$$R \text{ is difunctional } \equiv \langle \exists f,g \ : \ f \circ f^{\cup} = f{\prec} = g \circ g^{\cup} = g{\prec} \ : \ R = f^{\cup} \circ g \rangle \ . \quad \square$$

Symmetry places a major role in reasoning about difunctional relations. (Obviously, R is difunctional equivales $R^{\cup}$ is difunctional.) But our definition of "functional" is asymmetric and reflects a right-to-left bias in our interpretation of relations as having inputs and outputs.[8]

Theorem 26 is due to Riguet [17]; its proof is the subject of this section. For the moment, we note that, by combining Theorems 26 and 23, we have:

Theorem 27. *A relation R is difunctional if and only if it is the union of a set of completely disjoint rectangles.* $\square$

Later, we say that difunctional relations are "characterised" by a pair of functional relations and we refer to Theorem 26 as the *Characterisation Theorem*. The formal definition is as follows.

Definition 28. A *characterisation* is a pair of functional relations with the same target (but possibly different sources). A *minimal characterisation* is a pair of relations f and g with the same target such that

$$f \circ f^{\cup} = f{\prec} = g \circ g^{\cup} = g{\prec} \ .$$

That is, a minimal characterisation is a pair of functional relations with equal left domains. $\square$

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[6] See also Winter [23]. Winter does not use the name but does denote the "différence" of relation R by R^d as well as citing [18].

[7] In $R = f^{\cup} \circ g$, let f (resp. g) take inputs from axis A (resp. B) in Fig. 1. If the elements of A (resp. B) are listed so that all a (resp. b) that map to the same $f.a$ (resp. $g.b$) are listed contiguously, then the sides of each rectangle group together all as and bs such that $f.a = g.b$. If the order of such "clusters" is fixed according to some criterion, then the picture (Boolean matrix) is the same whatever functions are chosen satisfying Theorem 26.

[8] Jaoua et al. [13] choose a left-to-right interpretation: they use the term "deterministic" to mean $R^{\cup} \circ R \subseteq I$. Their formulation of Theorem 26 is correspondingly different.

The “minimality” requirement —the domain restrictions on f and g— may be omitted (“without loss of generality” in mathematical jargon). It is necessary, however, to establishing the “essential” uniqueness of the characterisation. (See Theorem 30.) Formally we have:

Lemma 29. *Suppose f and g are functional relations with the same target. Then*

$$f^{\cup}\circ g \;\;=\;\; (g{<}\circ f)^{\cup}\circ(f{<}\circ g) \;\;.$$

Moreover, $g{<}\circ f$ and $f{<}\circ g$ are functional relations and

$$(g{<}\circ f)\circ(g{<}\circ f)^{\cup} \;\;=\;\; (g{<}\circ f){<} \;\;=\;\; (f{<}\circ g)\circ(f{<}\circ g)^{\cup} \;\;=\;\; (f{<}\circ g){<} \;\;.$$

That is, the pair $g{<}\circ f$ and $f{<}\circ g$ is a minimal characterisation. □

The name “difunctional” is suggestive of Theorem 26; Riguet’s 1948 paper [16, Proposition 11] introduces the notion and gives a (natural-language-based) proof. Riguet’s 1950 paper [17] states that it is a generalisation of the theorem that a relation R is a partial equivalence relation iff $R=f^{\cup}\circ f$ for some functional relation f. Since then it appears to have become a folklore theorem. Hutton and Voermans [11, lemma 39], for example, state the theorem but do not provide a proof nor an attribution. The English text of [20, p.75] suggests that Schmidt and Ströhlein may be aware of the theorem but they also do not provide a proof. (They prove the easy “if” part of the theorem but not the converse; [20, Proposition 4.4.10] states that the characterisation “may be achieved *in essentially one* fashion” (their emphasis) but the accompanying proof actually establishes that the characterisation can be achieved in *at most* one fashion. That is, if such a characterisation exists, it is unique “up to a bijection”.)

A theme of this section is how to formalise different proofs of Theorem 26. One issue is whether or not the so-called “power transpose” of a relation, espoused by Freyd and Ščedrov [10] and Bird and De Moor [5], is sufficiently expressive. A second issue is the extent to which pointwise (as opposed to point-free) reasoning is desirable.

Section 4.2 sets the scene. The proof of Theorem 26 is an “if-and-only-if” proof and the section begins with the (trivial) proof of the “if” part. The main task is thus to give an explicit construction of a characterisation of a given difunction (the “only-if” part). A formal theorem —Theorem 30— states that although the details of the proof may be different, the constructed characterisations are formally equivalent (in a way made precise by the theorem). A very informal outline of several different ways of making the construction is then given.

The informal account in section 4.2 is made precise in sections 4.3.1 and 4.3.2; the former proves Theorem 26 by showing how to construct a set of “rectangles” that “covers” a given difunctional relation whilst the latter presents a construction in terms of the “power transpose” of the given relation. As already remarked —see Theorem 27— no matter how a characterisation is constructed, it defines a “completely disjoint covering” of the given difunction.

4.2. Different proofs, identical characterisations

The proof of Theorem 26 is by mutual implication. Follows-from is straightforward. Assume

$$\langle\exists f,g \,:\, f\circ f^{\cup} = f{<} = g\circ g^{\cup} = g{<} \,:\, R = f^{\cup}\circ g\rangle \;\;.$$

Then

$$\begin{array}{cl}
 & R\circ R^{\cup}\circ R \\
= & \{\quad \text{assumption and converse} \quad\} \\
 & f^{\cup}\circ g\circ g^{\cup}\circ f\circ f^{\cup}\circ g \\
= & \{\quad \text{assumption: } f\circ f^{\cup} = g{<} = g\circ g^{\cup} \quad\} \\
 & f^{\cup}\circ g{<}\circ g{<}\circ g \\
= & \{\quad g{<}\circ g = g\text{, and } R = f^{\cup}\circ g \quad\} \\
 & R \;\;.
\end{array}$$

The much more demanding task —which occupies all of subsection 4.3— is to establish the existence of a (minimal) characterisation of a given difunction. The theorem that there is *at most one* (up to isomorphism[9]) is the following.

[9] We write $R\cong S$ when relations R and S are isomorphic: see Definition A.17.)

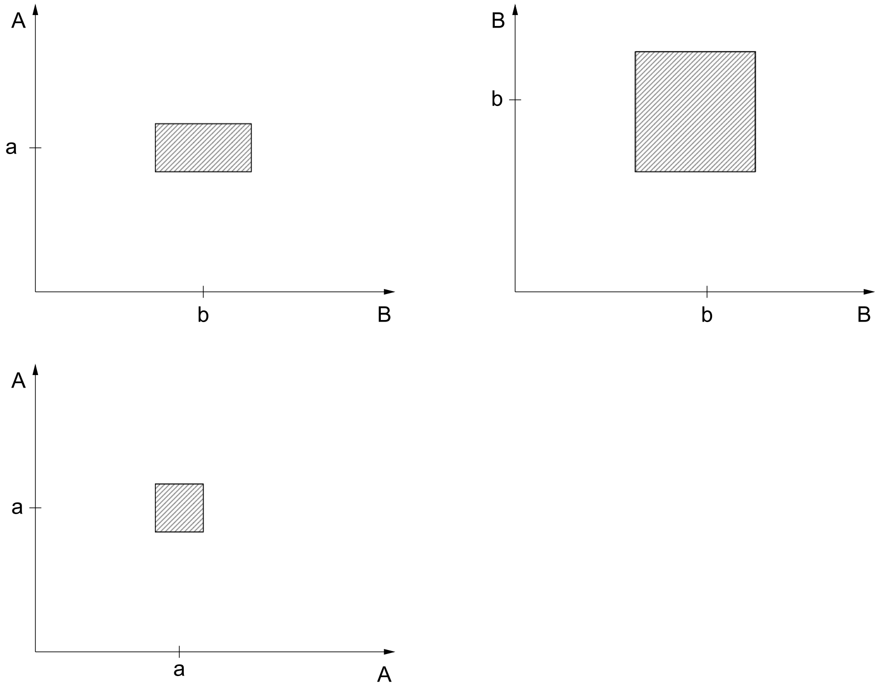

Fig. 2. Three different (but isomorphic) characterisations.

Theorem 30. *Suppose f and g are relations such that*

$$f \circ f^{\cup} = f{<} = g \circ g^{\cup} = g{<} \ .$$

Suppose also that h and k are relations such that

$$h \circ h^{\cup} = h{<} = k \circ k^{\cup} = k{<} \ .$$

Suppose further that

$$f^{\cup} \circ g = h^{\cup} \circ k \ .$$

Then

$$f \cong h \ \wedge \ g \cong k \ . \quad \square$$

As the name "functional" suggests, the only-if part of Theorem 26 is established by defining a type C, for each a in the left domain of R, a point $f.a$ in C, and, for each point b in the right domain of R, a point $g.b$ in C. The requirement is that, $f.a$ and $g.b$ are equal exactly when a and b are related by R. Fig. 2 shows three different but isomorphic (in the sense of Theorem 30) characterisations of the relation shown in Fig. 1.

In the top-left figure, the type C is the set of rectangles (relations of type $A{\sim}B$) defined by the relation R: the functional relation f maps a point a in the left domain of R to the rectangle defined by a and, similarly, the functional relation g maps a point b in the right domain of R to the rectangle defined by b. If a and b are points related by R, the rectangles $f.a$ and $g.b$ are equal; if a and b are not related by R, the rectangles $f.a$ and $g.b$ are not equal (and, in fact, they are "completely disjoint" in the sense that there are no points common to their sides).

In the top-right figure, the type C is a set of squares of type $B{\sim}B$ and, in the bottom-left figure the type C is a set of squares of type $A{\sim}A$. In the case of the top-right figure, the functional relation g maps point b to the square defined by b. The definition of f is more complicated: for a point a in the left domain of R, the value of $f.a$ is the square defined by some point b such that a and b are points related by R. The definitions of f and g are similar in the case of the bottom-left figure. (Just interchange the roles of a and b.)

Of course, a "square" is defined by a "side" of the square. So there is a fourth and a fifth way of representing a difunctional relation as a pair of functional relations: the type C can be defined to be the set of subsets of the left domain of R or the set of subsets of the right domain of R and, in each case, appropriate definitions of f and g must be constructed.

As mentioned earlier, all of these characterisations are the same — in the sense made precise by Theorem 30.

4.3. The characterisation theorem

As illustrated by Fig. 2, there are three different ways to approach the proof[10] of Theorem 26. The top-right and bottom-left figures are "dual" in the sense that one depicts a homogeneous relation on the target of the given relation whilst the other depicts a homogeneous relation on the source of the given relation. The top-left figure is more attractive because it does not exhibit any bias towards the source or target of the given relation. Section 4.3.1 presents such an unbiased proof of Theorem 26 whilst section 4.3.2 presents the dual proofs.

4.3.1. The rectangle proof

A relation R is a partial equivalence relation exactly when $R \circ R^{\cup} = R$; the "classes" of R are the squares $R \circ a \circ R^{\cup}$ where a is a point such that $a \subseteq R$. A relation R is a difunction exactly when $R \circ R^{\cup} \circ R = R$. By analogy and type considerations, this suggests that, if $a \subseteq R{<}$, the rectangle defined by a is given by $R \circ R^{\cup} \circ a \circ R$; similarly, if $b \subseteq R{>}$, the rectangle defined by b is given by $R \circ b \circ R^{\cup} \circ R$. The following lemma is the key to the proof.

Lemma 31. *Suppose R of type $A \sim B$ is difunctional. Then, for all points a and b,*

$$a \circ \top\!\!\top \circ b \subseteq R \ \Rightarrow\ R \circ R^{\cup} \circ a \circ R \ =\ R \circ b \circ R^{\cup} \circ a \circ R \ =\ R \circ b \circ R^{\cup} \circ R \ .$$

(Proof in Appendix B.) □

The "only-if" part of Theorem 26 is a consequence of Lemma 31. Specifically, suppose R is difunctional. Let C be the set of subsets of the relation R defined as follows:

$$C \ =\ \{a : a \subseteq R{<} : R \circ R^{\cup} \circ a \circ R\} \ .$$

(The dummy a ranges over points.) Note that $C = C'$ where

$$C' \ =\ \{b : b \subseteq R{>} : R \circ b \circ R^{\cup} \circ R\}$$

since

$$\begin{array}{ll}
 & \{a \ :\ a \subseteq R{<} \ :\ R \circ R^{\cup} \circ a \circ R\} \\
= & \qquad \{ \qquad \text{domains} \quad \} \\
 & \{a \ :\ \langle \exists b :: a \circ R \circ b = a \circ \top\!\!\top \circ b \rangle \ :\ R \circ R^{\cup} \circ a \circ R\} \\
= & \qquad \{ \qquad \text{range disjunction} \quad \} \\
 & \{a,b \ :\ a \circ R \circ b = a \circ \top\!\!\top \circ b \ :\ R \circ R^{\cup} \circ a \circ R\} \\
= & \qquad \{ \qquad \text{assumption: } R \text{ is difunctional; Lemma 31} \quad \} \\
 & \{a,b \ :\ a \circ R \circ b = a \circ \top\!\!\top \circ b \ :\ R \circ b \circ R^{\cup} \circ R\} \\
= & \qquad \{ \qquad \text{range disjunction and domains (as in first two steps)} \quad \} \\
 & \{b \ :\ b \subseteq R{>} \ :\ R \circ b \circ R^{\cup} \circ R\} \ .
\end{array}$$

Define f and g by, for all points a such that $a \subseteq R{<}$ and all points b such that $b \subseteq R{>}$,

$$f.a \ =\ R \circ R^{\cup} \circ a \circ R \quad \wedge \quad g.b \ =\ R \circ b \circ R^{\cup} \circ R \ . \tag{32}$$

Then, by definition, f and g are both functional, and surjective onto C and C', respectively. That is —exploiting the fact that C and C' are equal—

$$f \circ f^{\cup} = I_C = g \circ g^{\cup} \ .$$

We must now show that $R = f^{\cup} \circ g$. Guided by the definitions of f and g, we calculate that:

$$\begin{array}{ll}
 & R \circ R^{\cup} \circ a \circ R \ =\ R \circ b \circ R^{\cup} \circ R \\
\Rightarrow & \qquad \{ \qquad \text{Leibniz} \quad \} \\
 & R \circ R^{\cup} \circ a \circ R \circ R^{\cup} \ =\ R \circ b \circ R^{\cup} \circ R \circ R^{\cup}
\end{array}$$

[10] Strictly, the "only-if" part of the proof. Recall from section 4.2 that the "if" part is trivial.

$$
\begin{array}{ll}
\Rightarrow & \{ \quad \text{assumption: } R \text{ is difunctional (thus so too is } R^{\cup}\text{),} \\
& \quad\;\; R{<} \subseteq R\circ R^{\cup} \quad \} \\
& R{<}\circ a\circ R{<} \subseteq R\circ b\circ R^{\cup} \\
= & \{ \quad \text{assumption: } a \subseteq R{<} \quad \} \\
& a \subseteq R\circ b\circ R^{\cup} \\
= & \{ \quad \text{Lemma A.9} \quad \} \\
& a\circ\top\!\!\top\circ b \subseteq R \\
\Rightarrow & \{ \quad \text{assumption: } R \text{ is difunctional; Lemma 31} \quad \} \\
& R\circ R^{\cup}\circ a\circ R \;=\; R\circ b\circ R^{\cup}\circ R \;\; .
\end{array}
$$

We conclude (by mutual implication) that

$$R\circ R^{\cup}\circ a\circ R \;=\; R\circ b\circ R^{\cup}\circ R \;\;\equiv\;\; a\circ\top\!\!\top\circ b \subseteq R \;\; .$$

But, by the definitions of f and g and the definition of function application,

$$R\circ R^{\cup}\circ a\circ R \;=\; R\circ b\circ R^{\cup}\circ R \;\;\equiv\;\; a\circ\top\!\!\top\circ b \subseteq f^{\cup}\circ g \;\; .$$

Thus $R = f^{\cup}\circ g$ by the saturation axiom: (A.4).

4.3.2. The power-transpose construction

Recalling Fig. 2 once again, two alternative —but dual— ways of proving Theorem 26 are to construct functional relations that return square relations. Equivalently, one can construct functional relations that return the “side” of such a square, i.e. a subset of the source or, dually, a subset of the target of the given difunctional relation. In this section, we present such a construction using the power transpose function. The proof was obtained by revising the proof given by Jaoua et al. [13] in a way that eliminated the unnecessary assumption that R is homogeneous. One component of the characterisation is the relation $\Gamma R\circ R^{\cup}$. Since this is not obviously functional, we need a lemma to show that it is.

Lemma 33. *For all relations R,*

$$R \text{ is difunctional} \quad\equiv\quad \Gamma R\circ R^{\cup} \;\subseteq\; \Gamma(R\circ R^{\cup})\circ R{<} \;\; .$$

(Proof in Appendix B.) □

Corollary 34. *For all difunctional relations R,*

$$(\Gamma R\circ R^{\cup}) \,\circ\, (\Gamma R\circ R^{\cup})^{\cup} \;\;=\;\; \Gamma R \circ R{>}\circ(\Gamma R)^{\cup} \;\; .$$

In particular, if R is difunctional, $\Gamma R\circ R^{\cup}$ is functional.

(Proof in Appendix B.) □

Theorem 35. *Suppose R is a difunctional relation. Then the relations $\Gamma R \circ R^{\cup}$ and $\Gamma R \circ R{>}$ are both functional. Moreover,*

$$(\Gamma R\circ R^{\cup}) \circ (\Gamma R\circ R^{\cup})^{\cup} \;\;=\;\; (\Gamma R\circ R{>}) \circ (\Gamma R\circ R{>})^{\cup}$$

and

$$R \;\;=\;\; (\Gamma R\circ R^{\cup})^{\cup} \,\circ\, (\Gamma R\circ R{>}) \;\; .$$

That is, these two relations fulfil the requirements of f and g in Theorem 26.

Dually, the relations $\Gamma(R^{\cup})\circ R$ and $\Gamma(R^{\cup})\circ R{<}$ are both functional. Moreover,

$$(\Gamma(R^{\cup})\circ R{<}) \circ (\Gamma(R^{\cup})\circ R{<})^{\cup} \;\;=\;\; (\Gamma(R^{\cup})\circ R) \circ (\Gamma(R^{\cup})\circ R)^{\cup}$$

and

$$R \;\;=\;\; (\Gamma(R^{\cup})\circ R{<})^{\cup} \,\circ\, (\Gamma(R^{\cup})\circ R) \;\; .$$

That is, these two functions also fulfil the requirements of f and g in Theorem 26.

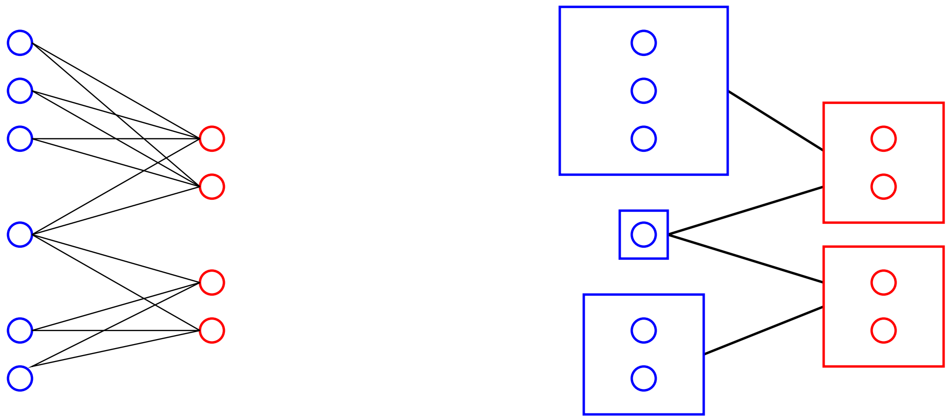

Fig. 3. A Relation and its core. (Colours are shown in the web version of this article.)

(Proof in Appendix B.) $\square$

An instance of Theorem 35 is that a relation R is a per iff

$$R \;=\; (\Gamma R \circ R{\succ})^{\cup} \circ (\Gamma R \circ R{\succ}) \;\;. \tag{36}$$

In order to show that this is the case, it is necessary to prove that, for a per R,

$$\Gamma R \circ R^{\cup} \;=\; \Gamma R \circ R{\succ} \;\;. \tag{37}$$

The proof of (37) can be found in Appendix B.

5. Reduction to the core

In this section, we introduce the notion of a "core" of an (arbitrary) relation. The main theorem is that a relation is difunctional iff its core is a (partial) bijection. See Theorem 44. Full proofs of the properties stated here can be found in [2]. A more detailed analysis of "cores" and "indexes" of a relation is currently in preparation [21].

Suppose R is an arbitrary relation. Both $R{\prec}$ and $R{\succ}$ are pers so can be characterised by their equivalence classes. Specifically, for a given R, suppose

$$R{\prec} = \lambda^{\cup} \circ \lambda \;\;\wedge\;\; R{\succ} = \rho^{\cup} \circ \rho$$

where λ and ρ are functional relations. Then

$$R = \lambda^{\cup} \circ \lambda \circ R \circ \rho^{\cup} \circ \rho \;\;.$$

The relation $\lambda \circ R \circ \rho^{\cup}$, which we denote by $|R|$, is a relation on the equivalence classes.

Definition 38 *(Core).* Suppose R is an arbitrary relation and suppose

$$R{\prec} = \lambda^{\cup} \circ \lambda \quad\wedge\quad R{\succ} = \rho^{\cup} \circ \rho$$

where λ and ρ are functional relations. Then the *core* of R, which is denoted by $|R|$, is defined by

$$|R| \;=\; \lambda \circ R \circ \rho^{\cup} \;\;. \quad \square$$

Example 39. Fig. 3 depicts a relation (on the left) and its core (on the right). Both are depicted as bipartite graphs. The relation R is a relation on blue and red nodes (the left and right columns, respectively). Its core $|R|$ is depicted as a relation on squares of blue nodes and squares of red nodes, each square being an equivalence class of $R{\prec}$ (on the left) or of $R{\succ}$ (on the right). $\square$

Lemma 40. *Suppose R, λ and ρ are as in Definition 38. Then*

$$R \;=\; \lambda^{\cup} \circ |R| \circ \rho \;\;. \quad \square$$

There are several different ways in which a per can be written as $f^{\cup} \circ f$ for some functional relation f. However, all are "isomorphic". Correspondingly, there are several different ways to construct a core of a relation, but all are "isomorphic" in the sense of Definition A.17.

A distinguishing feature of the core of a relation is that its left and right per-domains equal its left and right domains, respectively.

Theorem 41. *Suppose R, λ and ρ are as in Definition 38. Then*

$$|R|_{\succ} = |R|_{>} \ . \tag{42}$$

Also,

$$|R|_{\prec} = |R|_{<} \ . \quad \Box \tag{43}$$

(The proof of Theorem 41 has been omitted because it involves several non-trivial properties of factors whose inclusion would considerably lengthen this paper. Full details are given in [2].)

A general property of the core of a difunction is the following.

Theorem 44. *Suppose R is difunctional. Then the core of R is functional and injective. Specifically, if $R = f^{\cup} \circ g$ where $f \circ f^{\cup} = f_{<} = g \circ g^{\cup} = g_{<}$, then*

$$|R| \circ |R|^{\cup} = f_{<} \ \wedge \ |R|^{\cup} \circ |R| = g_{<} \ .$$

Thus, if R is difunctional, its core $|R|$ defines a (1–1) correspondence between the equivalence classes of $R_{\prec}$ and the equivalence classes of $R_{\succ}$.

(Proof in Appendix B.) $\Box$

A relation that is both injective and functional establishes a (1–1) correspondence between the points of its left and right domains. If these points are ordered arbitrarily but in such a way that the ordering respects the correspondence, and the relation is depicted by a graph whose axes depict the orderings of the domains, the relation will form a subdiagonal of the graph. Thus the mental picture of the core $|R|$ of a difunctional relation R is a subdiagonal of a graph; the mental picture of the (difunctional) relation R itself is a collection of completely disjoint rectangles arranged along the diagonal of a graph.

6. Conclusion

The notion of a difunctional relation is now generally attributed to Riguet [16]; Jaoua et al. [13] use the name "regular relation" but later publications [14] use the name "difunctional relation". The notions of a rectangle and completely disjoint rectangles, and elementary facts about difunctional relations, in particular Theorems 23 and 26, are discussed by Riguet [16].

Theorem 26 is also stated in [13, Proposition 4.12] and a proof given. Their proof assumes the relation is homogeneous; the proof of Theorem 35 is inspired by their proof whilst avoiding the assumption. Winter [23] assumes that a per can be characterised by a single functional relation and then uses this fact to prove Theorem 26.

His (very short and elegant) proof gives a different —albeit isomorphic— characterisation of a difunction. It is not clear whether or not Riguet was aware of such a construction although, in [17], he does state the characterisation of difunctions as a pair of functional relations. (Theorem 26 is a generalisation of the theorem that a partial equivalence relation is characterised by a single functional relation. See (36).)

Our contribution has been to compare different algebraic proofs of the theorem: point-free and pointwise proofs. The use of points requires the addition of several axioms to standard axiomatisations of (point-free) relation algebra, as briefly summarised in section 2, whereas the point-free calculations require only the existence of power transpose together with a minimal number of additional properties. Also, the point-free calculations combine concision with precision, which is an attribute that we value greatly. On the other hand, the point-free algebra is less expressive: it is difficult to see how Riguet's formulation of a difunction as a collection of completely disjoint rectangles would be expressed using just the notion of a power transpose. Perhaps surprisingly, our conclusion is that the pointwise proof is preferable to the proof that exploits a point-free characterisation of power transpose. This is because of the simplicity of the step from the elementary characterisation of difunctional relations (Theorem 25) to a set of rectangles ("réunions de rectangles"): recall section 4.3.1. The techniques used demonstrate that pointwise arguments can be economical whilst also increasing understanding.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Elements of relation algebra

Relation algebra comprises three layers with interfaces between the layers plus additional axioms peculiar to relations. (It is useful to separate the layers for use in other application areas.)

A (heterogeneous) relation of type $A{\sim}B$ is a subset of the cartesian product $A{\times}B$. So the first layer axiomatises the properties of the powerset $2^{A\times B}$. For this layer, we use the conventional notations of set theory, which we assume are familiar to the reader. However, we use $\perp\!\!\!\perp$ for the empty set (rather than the conventional $\emptyset$) and $\top\!\!\!\top$ for the full set $A{\times}B$. In keeping with the conventional practice of overloading the symbol "$\emptyset$", both these symbols are overloaded. The symbols "$\perp\!\!\!\perp$" and "$\top\!\!\!\top$" are pronounced "bottom" and "top", respectively. (Strictly we should write something like ${}_A\!\perp\!\!\!\perp_B$ and ${}_A\!\top\!\!\!\top_B$ for the bottom and top elements of type $A{\sim}B$. Of course, care needs to be taken when overloading operators in this way but it is usually the case that elementary type considerations allow the appropriate type to be deduced.)

The second layer adds a composition operator. If R is a relation of type $A{\sim}B$ and S is a relation of type $B{\sim}C$, the composition of R and S is a relation of type $A{\sim}C$ which we denote by $R{\circ}S$. Composition is associative and, for each type A, there is an identity relation which we denote by I_A. We often overload the notation for the identity relation, writing just I. Occasionally, for greater clarity, we do supply the type information.

The interface between the first and second layers defines a relation algebra to be an instance of a *regular algebra* [1] (also called a *standard Kleene algebra*, or **S**-*algebra* [8]). For this paper, the most important aspect of this interface is the existence and properties of the factor operators. These are introduced in section A.2. Also, $\perp\!\!\!\perp$ is a zero of composition: for all R, $\perp\!\!\!\perp{\circ}R = \perp\!\!\!\perp = R{\circ}\perp\!\!\!\perp$.

The third layer is the introduction of a *converse* operator. If R is a relation of type $A{\sim}B$, the converse of R, which we denote by $R^{\cup}$ (pronounced R "wok") is a relation of type $B{\sim}A$. The interface with the first layer is that converse is a poset isomorphism (in particular, $\perp\!\!\!\perp^{\cup} = \perp\!\!\!\perp$ and $\top\!\!\!\top^{\cup} = \top\!\!\!\top$), and the interface with the second layer is formed by the two rules $I^{\cup} = I$ and, for all relations R and S of appropriate type, $(R{\circ}S)^{\cup} = S^{\cup}{\circ}R^{\cup}$.

Additional axioms characterise properties peculiar to relations. An example is the so-called "modular law" (aka "Dedekind's rule" [16]). We don't need the modular law directly here; it is, however, necessary to the derivation of some of the properties we state without proof (for example, the properties of the domain operators given in section A.3). Another example is the "cone rule" (aka "Tarski's rule") which is used to prove (for example) Lemma 17.

A.1. Points and pairs

If A is a type, we use a, a' etc. to denote "points" of type A. Similarly for "points" of type B. "Points" represent elements of the appropriate type. Properties we use of a point a of type A are:

$$a{\circ}a \;=\; a \;=\; a^{\cup} \;\;, \tag{A.1}$$

$$\top\!\!\!\top{\circ}a{\circ}\top\!\!\!\top \;=\; \top\!\!\!\top \;\;, \tag{A.2}$$

$$a{\circ}\top\!\!\!\top{\circ}a \;=\; a \;\;, \tag{A.3}$$

$$\langle\forall p \;::\; p \subseteq I_A \;\equiv\; p = \langle\cup a : a \subseteq p : a\rangle\rangle \;\;. \tag{A.4}$$

Also, for points a and a' of the same type,

$$a = a' \;\vee\; a{\circ}a' = \perp\!\!\!\perp \;\;. \tag{A.5}$$

The *saturation* property is that

$$\langle\forall R \;::\; R = \langle\cup a,b : a{\circ}\top\!\!\!\top{\circ}b \subseteq R : a{\circ}\top\!\!\!\top{\circ}b\rangle\rangle \;\;. \tag{A.6}$$

The *irreducibility* property is that, if $\mathcal{R}$ is a function with range relations of type $A{\sim}B$ and source K, then, for all points a and b of appropriate type,

$$a{\circ}\top\!\!\!\top{\circ}b \subseteq \cup\mathcal{R} \;\equiv\; \langle\exists k : k{\in}K : a{\circ}\top\!\!\!\top{\circ}b \subseteq \mathcal{R}.k\rangle \;\;. \tag{A.7}$$

The *identity relation* I_A of type A has the property that, for all points a and a' of type A,

$$a{\circ}\top\!\!\!\top{\circ}a' \subseteq I_A \;\equiv\; a = a' \;\;. \tag{A.8}$$

For a point b, the square $R{\circ}b{\circ}R^{\cup}$ represents the set of all points a such that a and b are related by R. To be precise, for all relations R and points a and b,

$$(a \subseteq R{\circ}b{\circ}R^{\cup}) \;=\; (a{\circ}\top\!\!\!\top{\circ}b \subseteq R) \;=\; (b \subseteq R^{\cup}{\circ}a{\circ}R) \;\;. \tag{A.9}$$

A.2. Factors

In relation algebra, factors are also known as "residuals". We prefer the term "factor" because it emphasises calculational properties whereas "residual" emphasises an operational understanding (what is left after taking something away). In particular, factors have the *cancellation* properties:

$$T \circ T \backslash U \subseteq U \quad \wedge \quad R/S \circ S \subseteq R \ .$$

The factor operators (which we pronounce "under" and "over" respectively) are mutually associative. That is

$$R\backslash(S/T) = (R\backslash S)/T \ . \tag{A.10}$$

This means that it is unambiguous to write $R\backslash S/T$ — which we shall do in order to promote the associativity property by making its use invisible (in the same way that the use of the associativity of composition is made invisible).

A.3. The domain operators

The name "domain operator" is chosen because of the fundamental properties: for all R and all coreflexives p,

$$R = R \circ p \ \equiv \ R_{>} \subseteq p \tag{A.11}$$

and

$$R = p \circ R \ \equiv \ R_{<} \subseteq p \ . \tag{A.12}$$

The domain operators are lower adjoints in Galois connections. Specifically, for all relations R and coreflexives p,

$$R_{<} \subseteq p \equiv R \subseteq p \circ \top\!\!\top \tag{A.13}$$

and

$$R_{>} \subseteq p \equiv R \subseteq \top\!\!\top \circ p \ . \tag{A.14}$$

This observation is important because it has the immediate consequence that the domain operators are monotonic (with respect to the subset relation) and distribute through the union of relations. These properties are used frequently in this paper, sometimes without explicit mention.

The domain operators help to interpret factors in relation algebra. Specifically, for all relations R of type $A \sim C$ and S of type $B \sim C$ (for some A, B and C) and all points a of type A and b of type B,

$$a \circ \top\!\!\top \circ b \subseteq R/S \ \equiv \ (b \circ S)_{>} \subseteq (a \circ R)_{>} \ . \tag{A.15}$$

Dually, for all relations R of type $C \sim A$ and S of type $C \sim B$, and all points a of type A and b of type B,

$$a \circ \top\!\!\top \circ b \subseteq R\backslash S \ \equiv \ (R \circ a)_{<} \subseteq (S \circ b)_{<} \ . \tag{A.16}$$

A.4. Isomorphic relations

Several theorems we present "characterise" classes of relations in terms of functional relations. Typically these characterisations unique "up to isomorphism".

Definition A.17. Suppose R and S are two relations (not necessarily of the same type). Then we say that R and S are *isomorphic* and write $R \cong S$ iff

$$\begin{array}{ll} \langle \exists \phi, \psi & \\ \quad : & \phi \circ \phi^{\cup} = R_{<} \ \wedge \ \phi^{\cup} \circ \phi = S_{<} \ \wedge \ \psi \circ \psi^{\cup} = R_{>} \ \wedge \ \psi^{\cup} \circ \psi = S_{>} \\ \quad : & R = \phi \circ S \circ \psi^{\cup} \\ \rangle \ . \ \square & \end{array}$$

A.5. Pers and per domains

It is useful to record the left and right domains of the relation $R\backslash\!\backslash R\circ R{>}$:

Lemma A.18. *For all R,*

$$(R\backslash\!\backslash R\circ R{>}){>} \;=\; R{>} \;=\; (R{>}\circ R\backslash\!\backslash R){<} \;\;,$$
$$(R\backslash\!\backslash R\circ R{>}){<} \;=\; R{>} \;=\; (R{>}\circ R\backslash\!\backslash R){>} \;\;,$$
$$R\backslash\!\backslash R\circ R{>} \;=\; R{>}\circ R\backslash\!\backslash R\circ R{>} \;=\; R{>}\circ R\backslash\!\backslash R \;\;. \quad \square$$

Lemma A.18 has the consequence that $R{\succ}$ can be defined equivalently by the equation

$$R{\succ} \;=\; R\backslash\!\backslash R\circ R{>} \tag{A.19}$$

and, moreover,

$$(R{\succ}){<} \;=\; R{>} \;=\; (R{\succ}){>} \;\;. \tag{A.20}$$

Given relation R, the relation $R^{\cup}\circ R$ is symmetric but not necessarily transitive. However, it is an upper bound on the right per domain of R. That is,

$$R^{\cup}\circ R \supseteq R{\succ} \;\;. \tag{A.21}$$

(Proof in Appendix B.) Dually, of course, we have:

$$R\circ R^{\cup} \supseteq R{\prec} \;\;. \tag{A.22}$$

It is useful to investigate the circumstances in which the inclusions in (A.21) and (A.22) become equalities. (Note that the three boolean terms in Lemma A.23 below are equivalent definitions of a difunction: see Theorem 25.)

Lemma A.23. *For all relations R,*

$$(R{\prec} = R\circ R^{\cup}) \;=\; (R = R\circ R^{\cup}\circ R) \;=\; (R^{\cup}\circ R = R{\succ}) \;\;.$$

(As usual, we overload the equality symbol: its usage here alternates between equality of relations and equality of booleans. The continued equality should be read conjunctionally.)

(Proof in Appendix B.) $\square$

We are now in a position to extend [16, Corollaire, p. 134] from equivalence relations to pers.

Lemma A.24. *For all relations R, the following statements are all equivalent.*

(i) *R is a per (i.e. $R = R^{\cup} \wedge R\circ R \subseteq R$),*
(ii) *$R = R^{\cup}\circ R$,*
(iii) *$R = R{\prec}$,*
(iv) *$R = R{\succ}$.* $\square$

Finally:

Lemma A.25. *Suppose $\mathcal{R}$ is an indexed bag of rectangles. Then $\mathcal{R}$ is completely disjoint iff*

$$\begin{array}{ll} & \langle\forall j :: \mathcal{R}.j \neq \perp\!\!\perp\rangle \\ \wedge & \langle\forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)^{\cup}\circ\mathcal{R}.k = \perp\!\!\perp \;\wedge\; \mathcal{R}.j\circ(\mathcal{R}.k)^{\cup} = \perp\!\!\perp\rangle \;\;. \end{array}$$

Also, $\mathcal{R}$ is completely disjoint and injective —i.e. an indexed set— iff

$$\begin{array}{ll} & \langle\forall j :: \mathcal{R}.j \neq \perp\!\!\perp\rangle \\ \wedge & \langle\forall j,k \;::\; j \neq k \;\Rightarrow\; (\mathcal{R}.j)^{\cup}\circ\mathcal{R}.k = \perp\!\!\perp \;\wedge\; \mathcal{R}.j\circ(\mathcal{R}.k)^{\cup} = \perp\!\!\perp\rangle \;\;. \end{array}$$

(Proof in Appendix B.) $\square$

Appendix B. Proofs

Proof of Lemma 20 First note that

$$R{<}\cap S{<} = \perp\!\!\perp \;\equiv\; R{<}\circ S{<} = \perp\!\!\perp$$

since the intersection of coreflexives is the same as their composition. Then

$$
\begin{array}{ll}
& R{<}\circ S{<} = \perp\!\!\perp \\
\Rightarrow & \{ \quad \perp\!\!\perp \text{ is zero of composition } \} \\
& R^{\cup}\circ R{<}\circ S{<}\circ S = \perp\!\!\perp \\
= & \{ \quad \text{domains } \} \\
& R^{\cup}\circ S = \perp\!\!\perp \\
\Rightarrow & \{ \quad \perp\!\!\perp \text{ is zero of composition } \} \\
& R\circ R^{\cup}\circ S\circ S^{\cup} = \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity, } [\, R = \perp\!\!\perp \equiv R\subseteq \perp\!\!\perp \,] \text{ (applied twice) } \} \\
& (I\cap R\circ R^{\cup})\circ(I\cap S\circ S^{\cup}) = \perp\!\!\perp \\
= & \{ \quad \text{domains } \} \\
& R{<}\circ S{<} = \perp\!\!\perp \quad .
\end{array}
$$

The lemma follows by mutual implication.

Proof of Lemma 21 As remarked in Lemma 17, the relation $R\circ c\circ S$ is a rectangle, for all points c and all relations R and S; so this is also true of $f^{\cup}\circ c\circ g$. This collection of rectangles covers $f^{\cup}\circ g$ since

$$
\begin{array}{ll}
& f^{\cup}\circ g \\
= & \{ \quad g = g{<}\circ g \text{ and saturation axiom: (A.4) } \} \\
& f^{\cup}\circ\langle\cup c : c\subseteq g{<} : c\rangle\circ g \\
= & \{ \quad \text{distributivity } \} \\
& \langle\cup c : c\subseteq g{<} : f^{\cup}\circ c\circ g\rangle \quad .
\end{array}
$$

To show that the function $\langle c : c\subseteq g{<} : f^{\cup}\circ c\circ g\rangle$ is an indexed set of completely disjoint rectangles, we apply Lemma A.25. First, if $c\subseteq g{<}$, the rectangle $f^{\cup}\circ c\circ g$ is non-empty since

$$
\begin{array}{ll}
& f^{\cup}\circ c\circ g = \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity } \} \\
& (f^{\cup}\circ c\circ g){>} = \perp\!\!\perp \\
= & \{ \quad \text{domains } \} \\
& (f{<}\circ c\circ g){>} = \perp\!\!\perp \\
= & \{ \quad f{<} = g{<} \text{ and } c\subseteq g{<} \ \} \\
& (c\circ g){>} = \perp\!\!\perp \\
\Rightarrow & \{ \quad \text{monotonicity } \} \\
& ((c\circ g){>}\circ g^{\cup}){>} = \perp\!\!\perp \\
= & \{ \quad \text{domains } \} \\
& (c\circ g\circ g^{\cup}){>} = \perp\!\!\perp \\
= & \{ \quad g\circ g^{\cup} = g{<} \text{ and } c\subseteq g{<} \ \} \\
& c = \perp\!\!\perp
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \quad c \text{ is a point} \quad \} \\
 & \mathsf{false} \quad .
\end{array}
$$

That is,

$$\langle \forall c \ : \ c \subseteq g_{<} \ : \ f^{\cup} \circ c \circ g \neq \bot\!\!\bot \rangle \quad .$$

Also, assuming that $c \subseteq g_{<}$ and $c \neq c'$, we have:

$$
\begin{array}{ll}
 & (f^{\cup} \circ c \circ g)^{\cup} \circ (f^{\cup} \circ c' \circ g) \\
= & \{ \quad \text{distributivity, } c = c^{\cup} \quad \} \\
 & g^{\cup} \circ c \circ f \circ f^{\cup} \circ c' \circ g \\
= & \{ \quad \text{assumption:} \quad f \circ f^{\cup} = g_{<} \quad \} \\
 & g^{\cup} \circ c \circ g_{<} \circ c' \circ g \\
= & \{ \quad c \subseteq g_{<} \quad \} \\
 & g^{\cup} \circ c \circ c' \circ g \\
= & \{ \quad \text{assumption: } c \neq c', \text{ (A.5) with } a,a' := c,c' \quad \} \\
 & \bot\!\!\bot \quad .
\end{array}
$$

That is,

$$\langle \forall c,c' \ : c \subseteq g_{<} : c \neq c' \Rightarrow (f^{\cup} \circ c \circ g)^{\cup} \circ (f^{\cup} \circ c' \circ g) = \bot\!\!\bot \rangle \quad .$$

An almost identical argument shows that

$$\langle \forall c,c' \ : c \subseteq g_{<} : c \neq c' \Rightarrow (f^{\cup} \circ c \circ g) \circ (f^{\cup} \circ c' \circ g)^{\cup} = \bot\!\!\bot \rangle \quad .$$

Applying Lemma A.25 with $\mathcal{R} := \langle c : c \subseteq g_{<} : f^{\cup} \circ c \circ g \rangle$, properties (9), (10) and (10) establish that $f^{\cup} \circ g$ is indeed an indexed set of completely disjoint rectangles. $\square$

Proof of Lemma 22 Suppose $\mathcal{R}$ is a set of completely disjoint rectangles indexed by the set K. Suppose also that $R = \cup\mathcal{R}$. Define the relations f and g by, for all k in K and all points a such that $a \subseteq R_{<}$,

$$k \circ \top\!\!\top \circ a \subseteq f \ \ \equiv \ \ a \circ (\mathcal{R}.k)_{<} = a \quad , \tag{B.1}$$

and, for all k in $K$and all points b such that $b \subseteq R_{>}$

$$k \circ \top\!\!\top \circ b \subseteq g \ \ \equiv \ \ (\mathcal{R}.k)_{>} \circ b = b \quad . \tag{B.2}$$

Both f and g are functional. For example, here is the proof that f is functional: for all j and k in K,

$$
\begin{array}{ll}
 & j \circ \top\!\!\top \circ k \subseteq f \circ f^{\cup} \\
= & \{ \quad \text{saturation axiom: (A.4) and irreducibility: (A.7)} \quad \} \\
 & \langle \exists a :: j \circ \top\!\!\top \circ a \subseteq f \wedge a \circ \top\!\!\top \circ j \subseteq f^{\cup} \rangle \\
= & \{ \quad \text{(B.1) and converse} \quad \} \\
 & \langle \exists a :: a \circ (\mathcal{R}.j)_{<} = a \wedge a \circ (\mathcal{R}.k)_{<} = a \rangle \\
\Rightarrow & \{ \quad \text{coreflexives} \quad \} \\
 & (\mathcal{R}.j)_{<} \cap (\mathcal{R}.k)_{<} \neq \bot\!\!\bot \quad .
\end{array}
$$

So

$$
\begin{array}{ll}
 & j \circ \top\!\!\top \circ k \subseteq f \circ f^{\cup} \\
= & \{ \quad f \circ f^{\cup} \text{ is symmetric (i.e. } j \circ \top\!\!\top \circ k \subseteq f \circ f^{\cup} \ \equiv \ k \circ \top\!\!\top \circ j \subseteq f \circ f^{\cup}) \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & j\circ\top\!\!\top\circ k \subseteq f\circ f^{\cup} \;\wedge\; k\circ\top\!\!\top\circ j \subseteq f\circ f^{\cup} \\
\Rightarrow & \{ \quad \text{above (applied twice, once with } j{,}k{:=}k{,}j) \quad \} \\
 & (\mathcal{R}.j){<} \cap (\mathcal{R}.k){<} \neq \perp\!\!\perp \;\wedge\; (\mathcal{R}.k){<} \cap (\mathcal{R}.j){<} \neq \perp\!\!\perp \\
= & \{ \quad \mathcal{R} \text{ is a set of completely disjoint rectangles, Definition 19} \quad \} \\
 & j = k \quad .
\end{array}
$$

That is, by the saturation axiom and the definition of I_K, $f\circ f^{\cup} \subseteq I_K$.

Both f and g are also surjective. For suppose k is in K. Then

$$
\begin{array}{ll}
 & \mathsf{true} \\
= & \{ \quad \text{Definition 19 with } j{:=}k \quad \} \\
 & \mathcal{R}.k \neq \perp\!\!\perp \\
= & \{ \quad \text{saturation axiom: (A.4)} \quad \} \\
 & \langle \exists a :: a\circ(\mathcal{R}.k){<} = a \rangle \\
= & \{ \quad \text{(B.1)} \quad \} \\
 & \langle \exists a :: k\circ\top\!\!\top\circ a \subseteq f \rangle \\
\Rightarrow & \{ \quad a \text{ and } k \text{ are points, so } k = k\circ\top\!\!\top\circ k = k\circ\top\!\!\top\circ a\circ\top\!\!\top\circ k \quad \} \\
 & k \subseteq f\circ f^{\cup} \quad .
\end{array}
$$

That is, by the saturation axiom, $I_K \subseteq f\circ f^{\cup}$.

Combining the functionality of f with its surjectivity, we conclude that $f\circ f^{\cup} = I_K$. Similarly, $g\circ g^{\cup} = I_K$. So we have constructed relations f and g such that

$$
f\circ f^{\cup} \;=\; f{<} = I_K \;=\; g\circ g^{\cup} \;=\; g{<} \quad . \tag{B.3}
$$

We now have to show that $R = f^{\cup}\circ g$. A first step is to show that $f{>} = R{<}$ and $g{>} = R{>}$. We have, for all points a,

$$
\begin{array}{ll}
 & a \subseteq R{<} \\
= & \{ \quad R = \cup\mathcal{R} \quad \} \\
 & a \subseteq (\cup\mathcal{R}){<} \\
= & \{ \quad \text{distributivity} \quad \} \\
 & a \subseteq \langle \cup k :: (\mathcal{R}.k){<} \rangle \\
= & \{ \quad \text{irreducibility of points: (A.3) and (A.7)} \quad \} \\
 & \langle \exists k :: a \subseteq (\mathcal{R}.k){<} \rangle \\
= & \{ \quad \text{coreflexives} \quad \} \\
 & \langle \exists k :: a\circ(\mathcal{R}.k){<} = a \rangle \\
= & \{ \quad \text{(B.1)} \quad \} \\
 & \langle \exists k :: k\circ\top\!\!\top\circ a \subseteq f \rangle \\
= & \{ \quad \text{domains} \quad \} \\
 & a \subseteq f{<} \quad .
\end{array}
$$

We conclude by the saturation axiom (A.4) that $f{>} = R{<}$. Again, the property $g{>} = R{>}$ is proved similarly. It follows that

$$
\begin{array}{ll}
 & (f^{\cup}\circ g){>} \\
= & \{ \quad \text{domains} \quad \} \\
 & (f{<}\circ g){>} \\
= & \{ \quad \text{(B.3) (specifically, } f{<} = g{<}) \quad \}
\end{array}
$$

$$
\begin{array}{ll}
 & g{>} \\
= & \{ \quad \text{above} \quad \} \\
 & R{>} \quad .
\end{array}
$$

Similarly, $(f^{\cup}\circ g){<} = R{<}$. So, for all points a and b such that $a \subseteq R{<}$ and $b \subseteq R{>}$,

$$
\begin{array}{ll}
 & a\circ f^{\cup}\circ g\circ b \\
= & \{ \quad \text{saturation axiom: (A.4) and distributivity} \quad \} \\
 & \langle \cup k : k\subseteq f{<} \wedge k\subseteq g{<} : a\circ f^{\cup}\circ k\circ g\circ b\rangle \\
= & \{ \quad \text{(B.3)} \quad \} \\
 & \langle \cup k : k\in K : a\circ f^{\cup}\circ k\circ g\circ b\rangle \\
= & \{ \quad \text{composition of relations} \quad \} \\
 & \langle \cup k : a\circ\top\!\top\circ k\subseteq f^{\cup} \wedge k\circ\top\!\top\circ b\subseteq g : a\circ\top\!\top\circ k\circ k\circ\top\!\top\circ b\rangle \\
= & \{ \quad \text{assumption: } a\subseteq R{<} \text{ and } b\subseteq R{>}\text{; (B.1) and (B.2), and } k \text{ is a point} \quad \} \\
 & \langle \cup k \ : \ a\circ(\mathcal{R}.k){<} = a \ \wedge \ (\mathcal{R}.k){>}\circ b = b \ : \ a\circ\top\!\top\circ b\rangle \\
= & \{ \quad a \text{ is a point, so } a\circ(\mathcal{R}.k){<} = a \ \vee \ a\circ(\mathcal{R}.k){<} = \perp\!\!\perp \\
 & \quad\ \ b \text{ is a point, so } (\mathcal{R}.k){>}\circ b = b \ \vee \ (\mathcal{R}.k){>}\circ b = \perp\!\!\perp \\
 & \quad\ \ \text{range disjunction and } \perp\!\!\perp \text{ is least} \quad \} \\
 & \langle \cup k :: a\circ(\mathcal{R}.k){<}\circ\top\!\top\circ(\mathcal{R}.k){>}\circ b\rangle \\
= & \{ \quad \text{domains and } \mathcal{R}.k \text{ is a rectangle} \quad \} \\
 & \langle \cup k :: a\circ\mathcal{R}.k\circ b\rangle \\
= & \{ \quad R = \langle \cup k :: \mathcal{R}.k\rangle \text{ and distributivity} \quad \} \\
 & a\circ R\circ b \quad .
\end{array}
$$

We conclude that $R = f^{\cup}\circ g$ by the saturation property (A.6).

Proof of Lemma 31 Assume R is difunctional. Assume also that $a\circ\top\!\top\circ b \subseteq R$. Then

$$
\begin{array}{ll}
 & R\circ b\circ R^{\cup}\circ R \\
= & \{ \quad b \text{ is a point} \quad \} \\
 & R\circ b\circ b\circ R^{\cup}\circ R \\
\subseteq & \{ \quad \text{assumption: } a\circ\top\!\top\circ b \subseteq R \text{ , Lemma A.9} \quad \} \\
 & R\circ b\circ R^{\cup}\circ a\circ R\circ R^{\cup}\circ R \\
\subseteq & \{ \quad R \text{ is difunctional} \quad \} \\
 & R\circ b\circ R^{\cup}\circ a\circ R \quad .
\end{array}
$$

That is,

$$R\circ b\circ R^{\cup}\circ R \ \subseteq \ R\circ b\circ R^{\cup}\circ a\circ R \ .$$

By a symmetric argument

$$R\circ R^{\cup}\circ a\circ R \ \subseteq \ R\circ b\circ R^{\cup}\circ a\circ R \ .$$

But, since a is a point (and thus coreflexive),

$$R\circ b\circ R^{\cup}\circ a\circ R \ \subseteq \ R\circ b\circ R^{\cup}\circ R \ .$$

Symmetrically,

$$R\circ b\circ R^{\cup}\circ a\circ R \;\subseteq\; R\circ R^{\cup}\circ a\circ R \;\;.$$

The lemma follows by the anti-symmetry of equality.

Proof of Lemma 33

$$\Gamma R\circ R^{\cup} \;\subseteq\; \Gamma(R\circ R^{\cup})\circ R{\scriptstyle <}$$

$=\qquad\{\qquad$ domains (specifically, $R^{\cup}\circ R{\scriptstyle <} = R^{\cup}$) $\quad\}$

$$\Gamma R\circ R^{\cup} \;\subseteq\; \Gamma(R\circ R^{\cup})$$

$=\qquad\{\qquad$ ΓR is a total function; shunting rule: (7) $\quad\}$

$$R^{\cup} \;\subseteq\; (\Gamma R)^{\cup}\circ\Gamma(R\circ R^{\cup})$$

$=\qquad\{\qquad$ (11) $\quad\}$

$$R^{\cup} \;\subseteq\; R\backslash(R\circ R^{\cup}) \cap ((R\circ R^{\cup})\backslash R)^{\cup}$$

$=\qquad\{\qquad$ converse is an order isomorphism, factors $\quad\}$

$$R\circ R^{\cup}\subseteq R\circ R^{\cup} \;\;\wedge\;\; R\circ R^{\cup}\circ R\subseteq R$$

$=\qquad\{\qquad$ definition $\quad\}$

R is difunctional .

Proof of Corollary 34 The proof is by mutual inclusion. First, for all relations R,

$$(\Gamma R\circ R^{\cup}) \circ (\Gamma R\circ R^{\cup})^{\cup}$$

$=\qquad\{\qquad$ converse $\quad\}$

$$\Gamma R\circ R^{\cup}\circ R\circ(\Gamma R)^{\cup}$$

$\supseteq\qquad\{\qquad$ $R^{\cup}\circ R\supseteq R{\scriptstyle >}$, monotonicity $\quad\}$

$$\Gamma R\circ R{\scriptstyle >}\circ(\Gamma R)^{\cup} \;\;.$$

Second, for all difunctional relations R,

$$\Gamma R\circ R^{\cup}\circ R\circ(\Gamma R)^{\cup} \;\subseteq\; \Gamma R\circ R{\scriptstyle >}\circ(\Gamma R)^{\cup}$$

$\Leftarrow\qquad\{\qquad$ assumption: R is difunctional; Lemma 33 $\quad\}$

$$\Gamma(R\circ R^{\cup})\circ R{\scriptstyle <}\circ(\Gamma(R\circ R^{\cup}))^{\cup} \;\subseteq\; \Gamma R\circ R{\scriptstyle >}\circ(\Gamma R)^{\cup}$$

$=\qquad\{\qquad$ $\Gamma(R\circ R^{\cup})$ is a total function, shunting : (7) and (8) $\quad\}$

$$R{\scriptstyle <} \;\subseteq\; (\Gamma(R\circ R^{\cup}))^{\cup}\circ\Gamma R\circ R{\scriptstyle >}\circ(\Gamma R)^{\cup}\circ\Gamma(R\circ R^{\cup})$$

$=\qquad\{\qquad$ (12) $\quad\}$

$$R{\scriptstyle <} \;\subseteq\; (R\circ R^{\cup})\backslash\!\backslash R\circ R{\scriptstyle >}\circ((R\circ R^{\cup})\backslash\!\backslash R)^{\cup}$$

$\Leftarrow\qquad\{\qquad$ domains (specifically $R{\scriptstyle <}\subseteq R\circ R^{\cup}$ and $R = R\circ R{\scriptstyle >}$) $\quad\}$

$$R\circ R{\scriptstyle >}\circ R^{\cup} \;\subseteq\; (R\circ R^{\cup})\backslash\!\backslash R\circ R{\scriptstyle >}\circ((R\circ R^{\cup})\backslash\!\backslash R)^{\cup}$$

$\Leftarrow\qquad\{\qquad$ monotonicity and converse $\quad\}$

$$R \;\subseteq\; (R\circ R^{\cup})\backslash\!\backslash R$$

$=\qquad\{\qquad$ assumption: R is difunctional

as in last two steps of proof of Lemma 33 $\quad\}$

true .

Proof of Theorem 35 That $\Gamma R\circ R{\scriptstyle >}$ is functional is immediate from the fact that ΓR is a total function (by definition) and $R{\scriptstyle >}$ is a subset of the identity relation. That $\Gamma R\circ R^{\cup}$ is functional was shown in Corollary 34. It remains to prove the final equation.

$$(\Gamma R\circ R^{\cup})^{\cup} \circ (\Gamma R\circ R{\scriptstyle >})$$

$$
\begin{array}{ll}
= & \{ \quad \text{converse} \quad \} \\
& R \circ (\Gamma R)^{\cup} \circ \Gamma R \circ R{>} \\
= & \{ \quad (12) \quad \} \\
& R \circ R \backslash\!\!\backslash R \circ R{>} \\
= & \{ \quad (14) \quad \} \\
& R \circ R{>} \\
= & \{ \quad \text{domains} \quad \} \\
& R \quad .
\end{array}
$$

The dual theorem is obtained by instantiating R to $R^{\cup}$ (and noting that R is difunctional equivales $R^{\cup}$ is difunctional) and simplifying.

Proof of (37) This is done as follows:

$$
\begin{array}{ll}
& \Gamma R \circ R^{\cup} \;=\; \Gamma R \circ R{>} \\
= & \{ \quad R \text{ is a per, so } R^{\cup} = R \text{ and } R{>} \subseteq R \quad \} \\
& \Gamma R \circ R \;\subseteq\; \Gamma R \circ R{>} \\
\Leftarrow & \{ \quad \Gamma R \text{ is functional} \quad \} \\
& R \;\subseteq\; (\Gamma R)^{\cup} \circ \Gamma R \circ R{>} \\
= & \{ \quad (12) \quad \} \\
& R \;\subseteq\; R \backslash\!\!\backslash R \circ R{>} \\
= & \{ \quad (14) \text{ and Theorem A.24} \quad \} \\
& \mathsf{true} \quad .
\end{array}
$$

Proof of Theorem 44 If R is difunctional, the characterisation of difunctionals given by Theorem 26 allows us to assume that $R = f^{\cup} \circ g$ where $f \circ f^{\cup} = f{<} = g \circ g^{\cup} = g{<}$. Then

$$
R{\prec} \;=\; f^{\cup} \circ f \;=\; R \circ R^{\cup} \;\wedge\; R{\succ} \;=\; g^{\cup} \circ g \;=\; R^{\cup} \circ R \quad .
$$

So

$$
\begin{array}{ll}
& |R| \circ |R|^{\cup} \\
= & \{ \quad \text{Definition 38} \quad \} \\
& f \circ R \circ g^{\cup} \circ g \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad \text{Definition 38} \quad \} \\
& f \circ R \circ R{\succ} \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad \text{per domains: (16)} \quad \} \\
& f \circ R \circ R^{\cup} \circ f^{\cup} \\
= & \{ \quad f^{\cup} \circ f \;=\; R \circ R^{\cup} \quad \} \\
& f \circ f^{\cup} \circ f \circ f^{\cup} \\
= & \{ \quad f \circ f^{\cup} \;=\; f{<} \quad \} \\
& f{<} \quad .
\end{array}
$$

That is, $|R|$ is functional with left domain $f{<}$, (the coreflexive representation of) the set of equivalence classes of $R{\prec}$. By symmetry, $|R|$ is injective with right domain $g{<}$, (the coreflexive representation of) the set of equivalence classes of $R{\succ}$.

Proof of (A.20) We have:

$$
\begin{array}{ll}
 & R^{\cup}\circ R \supseteq R{>} \\
= & \{ \quad \text{definition: (14)} \quad \} \\
 & R^{\cup}\circ R \supseteq R{>}\circ R\backslash\!\!\backslash R \\
= & \{ \quad \text{cancellation: (13)} \quad \} \\
 & R^{\cup}\circ R\circ R\backslash\!\!\backslash R \supseteq R{>}\circ R\backslash\!\!\backslash R \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & R^{\cup}\circ R \supseteq R{>} \\
\Leftarrow & \{ \quad \text{definition} \quad \} \\
 & \mathsf{true}\ .
\end{array}
$$

Proof of Lemma A.23 We have:

$$
\begin{array}{ll}
 & R^{\cup}\circ R = R{>} \\
= & \{ \quad \text{(A.21) and anti-symmetry} \quad \} \\
 & R^{\cup}\circ R \subseteq R{>} \\
= & \{ \quad \text{definition: (14)} \quad \} \\
 & R^{\cup}\circ R \subseteq R{>}\circ R\backslash\!\!\backslash R \\
\Leftarrow & \{ \quad R{>}\circ R^{\cup} = R^{\cup} \text{ and monotonicity} \quad \} \\
 & R^{\cup}\circ R \subseteq R\backslash\!\!\backslash R \\
= & \{ \quad R^{\cup}\circ R \text{ is symmetric, } R\backslash\!\!\backslash R = R\backslash R\cap(R\backslash R)^{\cup} \quad \} \\
 & R^{\cup}\circ R \subseteq R\backslash R \\
\Leftarrow & \{ \quad \text{factors} \quad \} \\
 & R\circ R^{\cup}\circ R \subseteq R \\
\Leftarrow & \{ \quad \text{(16)} \quad \} \\
 & R^{\cup}\circ R = R{>}\ .
\end{array}
$$

We have thus proved (by mutual implication), that

$$(R\circ R^{\cup}\circ R \subseteq R) \ = \ (R^{\cup}\circ R = R{>})\ .$$

But,

$$
\begin{array}{ll}
 & R\circ R^{\cup}\circ R \subseteq R \\
= & \{ \quad R = R\circ R{>} \text{ and } R{>} \subseteq R^{\cup}\circ R\text{; monotonicity} \quad \} \\
 & R\circ R^{\cup}\circ R \subseteq R \ \wedge\ R\subseteq R\circ R^{\cup}\circ R \\
= & \{ \quad \text{anti-symmetry} \quad \} \\
 & R = R\circ R^{\cup}\circ R
\end{array}
$$

Combining the two calculations (using the transitivity of boolean equality),

$$(R = R\circ R^{\cup}\circ R) \ = \ (R^{\cup}\circ R = R{>})\ .$$

The dual property,

$$(R{<} = R\circ R^{\cup}) \ = \ (R = R\circ R^{\cup}\circ R)$$

follows by symmetry.

Proof of Lemma A.25

$$
\begin{array}{ll}
 & \mathcal{R} \text{ is completely disjoint} \\
= & \{ \quad \text{Definition 19} \quad \} \\
 & \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\equiv\; (\mathcal{R}.j)^{<} \cap (\mathcal{R}.k)^{<} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{>} \cap (\mathcal{R}.k)^{>} = \perp\!\!\perp \rangle \\
= & \{ \quad \text{mutual implication} \quad \} \\
 & \quad \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Leftarrow\; (\mathcal{R}.j)^{<} \cap (\mathcal{R}.k)^{<} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{>} \cap (\mathcal{R}.k)^{>} = \perp\!\!\perp \rangle \\
 & \wedge \; \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)^{<} \cap (\mathcal{R}.k)^{<} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{>} \cap (\mathcal{R}.k)^{>} = \perp\!\!\perp \rangle \\
= & \{ \quad \text{contrapositive; Lemma 20} \quad \} \\
 & \quad \langle \forall j,k \;::\; \mathcal{R}.j = \mathcal{R}.k \;\Rightarrow\; (\mathcal{R}.j)^{<} \cap (\mathcal{R}.k)^{<} \neq \perp\!\!\perp \vee (\mathcal{R}.j)^{>} \cap (\mathcal{R}.k)^{>} \neq \perp\!\!\perp \rangle \\
 & \wedge \; \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \perp\!\!\perp \rangle \\
= & \{ \quad \text{Leibniz, reflexivity of equality, idempotence of intersection} \quad \} \\
 & \quad \langle \forall j \;::\; (\mathcal{R}.j)^{<} \neq \perp\!\!\perp \vee (\mathcal{R}.j)^{>} \neq \perp\!\!\perp \rangle \\
 & \wedge \; \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \perp\!\!\perp \rangle \\
= & \{ \quad \text{domains} \\
 & \qquad (\, [\, (R^{<} = \perp\!\!\perp) = (R = \perp\!\!\perp) = (R^{>} = \perp\!\!\perp) \,] \text{ with } R := \mathcal{R}.j)) \quad \} \\
 & \quad \langle \forall j \;::\; \mathcal{R}.j \neq \perp\!\!\perp \rangle \\
 & \wedge \; \langle \forall j,k \;::\; \mathcal{R}.j \neq \mathcal{R}.k \;\Rightarrow\; \mathcal{R}.j \circ (\mathcal{R}.k)^{\cup} = \perp\!\!\perp \wedge (\mathcal{R}.j)^{\cup} \circ \mathcal{R}.k = \perp\!\!\perp \rangle \;.
\end{array}
$$

Injectivity of $\mathcal{R}$ is the property that $\langle \forall j,k :: \mathcal{R}.j = \mathcal{R}.k \;\equiv\; j = k \rangle$. The characterisation of completely disjoint and injective thus follows by the use of Leibniz's rule.

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