The Thins Ordering on Relations

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Abstract

Earlier papers [BV22, BV23b, BV23a] introduced the notions of a core and an index of a relation (an index being a special case of a core). A limited form of the axiom of choice was postulated —specifically that all partial equivalence relations (pers) have an index— and the consequences of adding the axiom to axiom systems for point-free reasoning were explored. In this paper, we define a partial ordering on relations, which we call the thins ordering. We show that our axiom of choice is equivalent to the property that core relations are the minimal elements of the thins ordering. We also characterise the relations that are maximal with respect to the thins ordering. Apart from our axiom of choice, the axiom system we employ is paired to a bare minimum and admits many models other than concrete relations — we do not assume, for example, the existence of complements; in the case of concrete relations, the theorem is that the maximal elements of the thins ordering are the empty relation and the equivalence relations. This and other properties of thins provide further evidence that our axiom of choice is a desirable means of strengthening point-free reasoning on relations.

1 Introduction

Earlier papers [BV22, BV23b, BV23a] introduced the notions of a core and an index of a relation (an index being a special case of a core). In [BV23b] the focus was on strengthening standard axiom systems for point-free reasoning. A limited form of the axiom of choice was postulated —specifically that all partial equivalence relations (pers) have an index— and the consequences of adding the axiom were explored. The working document [BV22] extends this work to practical applications of the notions, an extract of which being [BV23a] on diagonals and block-ordered relations.

In this paper, we define a partial ordering on relations, which we call the thins ordering. We begin by defining thins on partial equivalence relations (pers), and then extend the ordering to all relations. We show that our axiom of choice is equivalent to the property that the minimal elements of the thins relation on pers are precisely the indexes of pers. (See theorem 24 for a precise statement.) We then extend the thins ordering to all relations and we show that, assuming our axiom of choice, the minimal elements of the ordering are precisely the

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core relations. (See theorem 64.) We also show that, when the thins relation is restricted to pers, equivalence relations are maximal. Our calculations make use of a limited subset of the axioms of point-free relation algebra —for example, we do not assume the existence of complements— thus testifying to the power of our axiom of choice.

Because this paper is an extension of [BV23b] we have omitted all introductory material. For ease of reference, we do repeat some key topics from [BV23b]. In such cases, we omit proofs of lemmas and theorems. Hints in our calculations often refer to properties proved in earlier publications; in such cases we state the properties within square brackets. (See for example the proof of lemma 8 where the hint in the first step is

$$[P^{\cup} \circ P = P = P \circ P^{\cup}] \quad .$$

The square brackets should be read as "everywhere". So the stated property is true for all instances of the dummy P, which ranges in this case over pers.) Nevertheless, [BV23b] is recommended reading before embarking on the current paper.

Section 2 gives a brief summary of our axiom system. The novel contributions of the paper begin in section 3 with the definition of the thins relation. (At this stage, we don't call it an "ordering" because that property has yet to be established.) We also reproduce the definition of an index and the axiom of choice from [BV23b].

Section 4 formulates a number of properties of thins. An important property (specifically, theorem 14) is that the indexes of a per Q are the pers P that thin Q and are coreflexive. This section also includes the proof that the thins relation is an ordering relation on pers.

Section 5 is about pers that are minimal with respect to the thins ordering. We show that coreflexive relations are minimal. The converse property —minimal implies coreflexive— is then shown to be equivalent to our axiom of choice. The conclusion of the section, theorem 24, is that the axiom of choice is equivalent to the conjunction of two properties: firstly, the minimal elements of the thins ordering on pers are precisely the coreflexive relations and, secondly, every per thins to a minimal element.

Section 6 is about maximality. A brief, informal summary of the main theorem (theorem 52) is that the equivalence relations are maximal with respect to the thins ordering on non-empty pers. The theorem we formulate is, in fact, more general than this since it applies to models of point-free relation algebra quite different from the standard set-theoretic binary relations. The statement of the theorem introduces a new idiom to point-free relation algebra that avoids a case analysis on whether or not a relation is empty.

Section 7 is where we extend the thins ordering to arbitrary relations (and not just pers). We prove that, assuming our axiom of choice, a relation S is minimal with respect to the thins ordering on arbitrary relations iff S is a core relation.

2 Point-Free Relation Algebras

In this section, we define a *point-free relation algebra*. Such an algebra has three components with interfaces between them: a (typed) monoid structure, a lattice structure, and a converse structure.

Underpinning any relation algebra is a very simple type structure. We assume the existence of a non-empty set $\mathcal{T}$ of so-called *basic types*. A *relation type* is an ordered pair of basic types. We write $A\sim B$ for the ordered pair of basic types A and B. We often omit "relation" and refer to $A\sim B$ as a "type". The carrier set of a point-free relation algebra is typed in the sense that each element X of the carrier set has a type $A\sim B$ for some basic types A and B. A relation of type $A\sim A$, for some A, is said to be *homogeneous*. If $\mathcal{T}$ has exactly one element we say that the algebra is *untyped*.

The monoid structure is defined as follows. For each triple of basic types A, B and C, and each element X of type $A\sim B$ and each element Y of type $B\sim C$, there is an element $X\circ Y$ of type $A\sim C$. Also for each basic type A there is an element $\mathbb{I}$ of type $A\sim A$. The element $X\circ Y$ is called the *composition* of X and Y, and $\mathbb{I}$ is called the *identity* of A. Composition is required to be associative, and identities are required to be the units of composition. The composition $X\circ Y$ is only defined when X and Y have appropriate types. (Such a typed monoid structure is commonly called a "category".)

In principle, the type of the identities should be made explicit in the notation we use: for example, by writing $\mathbb{I}_A$ for the identity of type $A\sim A$. It is convenient for us not to do so, leaving the type information to be deduced from the context. This is also the case for other operators and constants that we introduce below.

For each type $A\sim B$, we assume the existence of a (finitely) distributive lattice (partially) ordered by $\subseteq$. The binary supremum and infimum operators of the lattice are denoted by $\cup$ and $\cap$, respectively. The least and greatest elements of the lattice are denoted by $\perp\!\!\!\perp$ and $\top\!\!\!\top$, respectively.

The interface between the monoid structure and the lattice structure is the existence of the two factor operators defined by, for all X, Y and Z of appropriate types,

$$(Y\subseteq X\backslash Z) \;=\; (X\circ Y\subseteq Z) \;=\; (X\subseteq Z/Y) \quad .$$

As a consequence, composition distributes over supremum. That is, for all relations R, S, T and U of appropriate type

$$R\circ(S\cup T) \;=\; R\circ S\cup R\circ T \quad ,$$

$$(S\cup T)\circ U \;=\; S\circ U\cup T\circ U \quad ,$$

$$R\circ\perp\!\!\!\perp \;=\; \perp\!\!\!\perp \quad \text{, and}$$

$$\perp\!\!\!\perp\circ R \;=\; \perp\!\!\!\perp \quad .$$

(The symbol $\perp\!\!\!\perp$ is overloaded in the final two equations: each occurrence may have a different type.)

The converse structure is very simple: for each element X of type $A\sim B$ there is an element $X^{\cup}$ of type $B\sim A$.

The interface between the lattice structure and the converse structure is the Galois connection: for all X and Y of appropriate types,

$$X^{\cup}\subseteq Y \;\equiv\; X\subseteq Y^{\cup} \quad .$$

The interface between the monoid structure and the converse structure is: for each identity $\mathbb{I}$,

$$\mathbb{I}^{\cup}=\mathbb{I}$$

and, for all X and Y of appropriate types,

$$(X \circ Y)^{\cup} \;=\; Y^{\cup} \circ X^{\cup} \quad .$$

Finally, the *modularity law* acts as an interface between all three components: for all X, Y and Z of appropriate types,

$$X \circ Y \cap Z \subseteq X \circ (Y \cap X^{\cup} \circ Z) \quad .$$

We do not use the existence of the factor operators or the modularity law anywhere explicitly in this paper. We do, however, make extensive use of the properties of coreflexive relations and the (coreflexive-)domain operators first mentioned in section 3, as well as the per-domain operators in section 7. The properties of coreflexive domains rely heavily on the modularity law, and per domains are defined in terms of the factor operators.

A consequence of the axioms is that all the operators of the algebra —composition, converse, supremum, infimum and the domain operators introduced later— are monotonic with respect to the $\subseteq$ ordering. We exploit monotonicity frequently in our calculations, sometimes without explicit mention.

The axioms of point-free relation algebra do not completely characterise all the properties of binary relations and, therefore, admit other models (for example geometric models: see [Fv90, 2.158] and [Voe99, section 3.5]). We use the term *concrete relation* below to refer to binary relations as they are normally understood. That is, a *concrete relation of type* $A \sim B$ is an element of the powerset $2^{A \times B}$. (The types A and B do not need to be finite.)

A point-free relation algebra is said to be *unary* if it satisfies the *cone rule*: for all X,

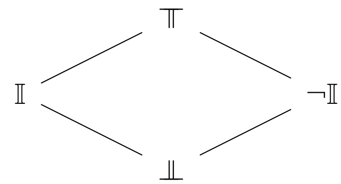
$$\mathbb{T} \circ X \circ \mathbb{T} = \mathbb{T} \quad \equiv \quad X \neq \mathbb{L} \quad .$$

(The three occurrences of $\mathbb{T}$ may have different types. The rightmost occurrence is assumed to have the same type as X; the other two are assumed to be homogeneous relations of the appropriate types. The terminology reflects the fact that the cartesian product of two relation algebras is non-unary. See [Voe99, section 3.4.3].)

In [BV22, BV23b], much of the focus was on introducing axioms that facilitate pointwise reasoning. To this end, the cone rule was used extensively. In contrast, in this paper the goal is not to facilitate pointwise reasoning but, instead, to strengthen point-free reasoning. So here the cone rule is deemed to be invalid. See the introductory remarks in section 6.

Example 1 The simplest examples of point-free relation algebras are all untyped. The simplest of all has just one element: all of the constants $\mathbb{L}$, $\mathbb{I}$ and $\mathbb{T}$ are defined to be equal. The second simplest has two elements $\mathbb{L}$ and $\mathbb{T}$; $\mathbb{I}$ is defined to be equal to $\mathbb{T}$ (and different from $\mathbb{L}$). The third simplest has three elements: the constants $\mathbb{L}$, $\mathbb{I}$ and $\mathbb{T}$, which are defined to be distinct. (In all three cases, the definitions of the ordering relation, composition and converse can be deduced from the axioms.)

A four-element algebra is obtained by adding a new element $\neg\mathbb{I}$ to the three-element algebra and defining the composition $\neg\mathbb{I} \circ \neg\mathbb{I}$ to be $\mathbb{I}$ and the converse $(\neg\mathbb{I})^{\cup}$ to be $\neg\mathbb{I}$. As suggested by the notation, $\neg\mathbb{I}$ is the complement of $\mathbb{I}$. That is, the lattice structure is as shown in the diagram below.



The simplest example is not unary, the other examples are unary. A model of the two-element algebra is formed by the (homogeneous) concrete relations on a set with exactly one element. The other examples do not have such a model since the concrete relations on a set of size n form a power set of size $2^{n\times n}$.
□

3 Basic Definitions

We begin by restricting our study to partial equivalence relations (pers[1]). In this section we recall the definition of an index of a per and our axiom of choice. New is definition 5.

Throughout the paper, P and Q denote pers. For pers, the left and right domains coincide. (I.e. for all pers P, $P{<} = P{>}$.) For this reason, $P{\asymp}$ is used to denote the left/right domain of P. That is, $P{<} = P{\asymp} = P{>}$. (We assume familiarity with the properties of coreflexive[2] relations and the domain operators. So, rather than include an extensive list of their properties, we state the properties being used between square "everywhere" brackets, as explained earlier.)

Definition 2 (Index of a Per) Suppose P is a per. Then an *index* of P is a relation J such that

(a) $J \subseteq P{\asymp}$,

(b) $J{\circ}P{\circ}J = J$,

(c) $P{\circ}J{\circ}P = P$.
□

Axiom 3 (Axiom of Choice) Every per has an index.
□

Example 4 The three- and four-element algebras detailed in example 1 do not satisfy the axiom of choice since, in both cases, $\top\!\!\top$ does not have an index. The two simplest examples do satisfy the axiom of choice because each element is an index of itself.
□

Definition 5 (Thins) The thins relation on pers is defined by, for all pers P and Q,

$$P \preceq Q \quad \equiv \quad P = P{\asymp} \circ Q \circ P{\asymp} \ \wedge \ Q = Q \circ P{\asymp} \circ Q \ .$$
□

[1]Relation P is a per iff it is symmetric (i.e. $P = P^{\cup}$) and transitive (i.e. $P{\circ}P \subseteq P$). Equivalently, P is a per iff $P = P \circ P^{\cup}$.

[2]Relation p is coreflexive iff $p \subseteq \mathbb{I}$.

This paper is about the properties of the $\preceq$ relation. We call it the thins relation. (So $P \preceq Q$ is pronounced P thins Q.) Much of the paper is about the thins relation on pers but we extend it to all relations in section 7.

Informally, the first conjunct in the definition of $P \preceq Q$ states that the equivalence classes of P are subsets of the equivalence classes of Q, and the second conjunct states that, for each equivalence class of P, there is a corresponding equivalence class of Q. Some of the properties stated below are intended to confirm this informal interpretation of the definition.

Example 6 In all four example algebras detailed in example 1, the pers are $\perp\!\!\!\perp$, $\mathbb{I}$ and $\top\!\!\!\top$ and the thins relation is discrete. (That is, the thins relation is the equality relation on the pers.)
□

4 Basic Properties

As the title suggests, this section is about basic properties of the thins relation. Theorem 9 establishes that it is a partial ordering on pers. Theorem 14 formulates an alternative definition of an index of a per in terms of thins. Subsequent lemmas anticipate properties needed in later sections.

Obvious from the property that $P_{\lessgtr} \subseteq \mathbb{I}$ (for arbitrary P) and monotonicity of composition, applied to the property $P = P_{\lessgtr} \circ Q \circ P_{\lessgtr}$ is that, for all pers P and Q,

(7) $\qquad P \preceq Q \;\Rightarrow\; P \subseteq Q$.

We use this frequently below.

The following lemma is also used on several occasions. Compare the lemma with properties 2(b) and 2(c) of an index.

Lemma 8 For all pers P and Q,

$$P \preceq Q \quad\Rightarrow\quad P = P{\circ}Q{\circ}P \;\wedge\; Q = Q{\circ}P{\circ}Q \quad .$$

Proof Assume $P = P_{\lessgtr} \circ Q \circ P_{\lessgtr}$. Then

$$\begin{array}{cl}
 & P{\circ}Q{\circ}P \\
= & \quad\{ \quad \text{domains (specifically } [\, P_{\lessgtr} \circ P = P = P \circ P_{\lessgtr} \,]) \quad \} \\
 & P \circ P_{\lessgtr} \circ Q \circ P_{\lessgtr} \circ P \\
= & \quad\{ \quad \text{assumption: } P = P_{\lessgtr} \circ Q \circ P_{\lessgtr} \quad \} \\
 & P{\circ}P{\circ}P \\
= & \quad\{ \quad P \text{ is a per, } [\, P = P{\circ}P \,] \quad \} \\
 & P \quad .
\end{array}$$

Since $P \preceq Q \;\Rightarrow\; P = P_{\lessgtr} \circ Q \circ P_{\lessgtr}$, it follows that $P \preceq Q \;\Rightarrow\; P = P{\circ}Q{\circ}P$. Also,

$$\begin{array}{cl}
 & Q{\circ}P{\circ}Q \\
\supseteq & \quad\{ \quad [\, P \supseteq P_{\lessgtr} \,] \quad \} \\
 & Q \circ P_{\lessgtr} \circ Q
\end{array}$$

$$
\begin{array}{ll}
= & \{ \quad \text{assumption: } P \preceq Q \text{, so } Q = Q \circ P_{\asymp} \circ Q \quad \} \\
 & Q \\
= & \{ \quad Q \text{ is a per, } [\, P = P \circ P \,] \text{ with } P := Q \quad \} \\
 & Q \circ Q \circ Q \\
\supseteq & \{ \quad \text{(7) and monotonicity} \quad \} \\
 & Q \circ P \circ Q \ .
\end{array}
$$

That is, by anti-symmetry, $P \preceq Q \ \Rightarrow \ Q = Q \circ P \circ Q$.
□

The notation we have chosen suggests that thins is a partial ordering. This is indeed the case:

Theorem 9 The thins relation is a partial ordering on pers.

Proof We must prove that the thins relation is reflexive, transitive and anti-symmetric. Reflexivity is straightforward:

$$
\begin{array}{ll}
 & P \preceq P \\
= & \{ \quad \text{definition 5} \quad \} \\
 & P = P_{\asymp} \circ P \circ P_{\asymp} \ \wedge \ P = P \circ P_{\asymp} \circ P \\
= & \{ \quad \text{domains (specifically, } [\, P_{\asymp} \circ P = P = P \circ P_{\asymp} \,]) \quad \} \\
 & P = P \ \wedge \ P = P \circ P \\
= & \{ \quad \text{reflexivitiy of equality; } P \text{ is a per and } [\, P = P \circ P \,] \quad \} \\
 & \text{true} \ .
\end{array}
$$

Now, suppose P, Q and R are pers, and $P \preceq Q$ and $Q \preceq R$. Applying (7), we have:

(10) $\quad P \subseteq Q \subseteq R$.

To prove transitivity, we must prove that $P \preceq R$. Applying definition 5, we must prove that

(11) $\quad P = P_{\asymp} \circ R \circ P_{\asymp}$

and

(12) $\quad R = R \circ P_{\asymp} \circ R$.

We prove (11) by mutual inclusion:

$$
\begin{array}{ll}
 & P \\
= & \{ \quad \text{domains (specifically, } [\, P_{\asymp} \circ P = P = P \circ P_{\asymp} \,]) \quad \} \\
 & P_{\asymp} \circ P \circ P_{\asymp} \\
\subseteq & \{ \quad \text{by (10), } P \subseteq R \text{ ; monotonicity} \quad \} \\
 & P_{\asymp} \circ R \circ P_{\asymp} \\
\subseteq & \{ \quad P \text{ is a per and } [\, P_{\asymp} \subseteq P \,] \text{ ; monotonicity} \quad \} \\
 & P \circ R \circ P \\
= & \{ \quad \text{assumption: } P = P_{\asymp} \circ Q \circ P_{\asymp} \quad \} \\
 & P_{\asymp} \circ Q \circ P_{\asymp} \circ R \circ P_{\asymp} \circ Q \circ P_{\asymp}
\end{array}
$$

$$
\begin{array}{ll}
\subseteq & \{\quad [\,P_{\asymp} \subseteq \mathbb{I}\,]\text{; monotonicity} \quad\} \\
& P_{\asymp} \circ Q \circ R \circ Q \circ P_{\asymp} \\
= & \{\quad \text{assumption: } Q \preceq R\text{, lemma 8 with } P,Q := Q,R \quad\} \\
& P_{\asymp} \circ Q \circ P_{\asymp} \\
= & \{\quad \text{assumption: } P = P_{\asymp} \circ Q \circ P_{\asymp} \quad\} \\
& P \quad .
\end{array}
$$

Now we prove (12). Again, the proof is by mutual inclusion:

$$
\begin{array}{ll}
& R \circ P_{\asymp} \circ R \\
\subseteq & \{\quad [\,P_{\asymp} \subseteq \mathbb{I}\,] \text{ and monotonicity} \quad\} \\
& R \circ R \\
\subseteq & \{\quad R \text{ is a per (and hence transitive)} \quad\} \\
& R \\
= & \{\quad \text{assumption: } Q \preceq R\text{, lemma 8 with } P,Q := Q,R \quad\} \\
& R \circ Q \circ R \\
= & \{\quad \text{assumption: } P \preceq Q \quad\} \\
& R \circ Q \circ P_{\asymp} \circ Q \circ R \\
\subseteq & \{\quad \text{assumption: } Q \preceq R\text{, (7) and monotonicity} \quad\} \\
& R \circ R \circ P_{\asymp} \circ R \circ R \\
\subseteq & \{\quad R \text{ is a per (and hence transitive)} \quad\} \\
& R \circ P_{\asymp} \circ R \quad .
\end{array}
$$

Finally, combining (11) and (12) and applying definition 5 (with $P,Q := P,R$) we have shown that $P \preceq R$. This concludes the proof that the thins relation is transitive.

Finally, we prove that the thins relation is anti-symmetric. Suppose $P \preceq Q \preceq P$. Then, by (7), $P \subseteq Q \subseteq P$. Thus $P = Q$ by the anti-symmetry of the $\subseteq$ relation.

□

We now consider the properties of indexes with respect to the thins ordering. Our first goal is to show that the definition of an index J of a per P can be split into two conjuncts, namely $J \subseteq \mathbb{I}$ and $J \preceq P$. (See theorem 14.) First, a lemma:

Lemma 13 If J is an index of P then $J \preceq P$.

Proof

$$
\begin{array}{ll}
& J \text{ is an index of } P \\
= & \{\quad \text{definition 2} \quad\} \\
& J \subseteq P_{\asymp} \;\wedge\; J \circ P \circ J = J \;\wedge\; P \circ J \circ P = P \\
\Rightarrow & \{\quad \text{domains (specifically, } [\,P_{\asymp} \subseteq \mathbb{I}\,] \text{ and } [\; Q \subseteq \mathbb{I} \equiv Q = Q_{\asymp}\;] \text{ with } Q := J) \quad\} \\
& J = J_{\asymp} \;\wedge\; J \circ P \circ J = J \;\wedge\; P \circ J \circ P = P \\
\Rightarrow & \{\quad \text{Leibniz} \quad\} \\
& J_{\asymp} \circ P \circ J_{\asymp} = J \;\wedge\; P \circ J_{\asymp} \circ P = P \\
= & \{\quad \text{definition 5} \quad\} \\
& J \preceq P \quad .
\end{array}
$$

□

Theorem 14 For all pers P and Q,

$$P \subseteq \mathbb{I} \ \wedge \ P \preceq Q \quad \equiv \quad P \text{ is an index of } Q \ .$$

Proof The proof is by mutual implication. For ease of reference, we instantiate definition 2 with $J,P := P,Q$:

$$(15) \qquad P \subseteq Q{\scriptstyle\asymp} \ \wedge \ P{\circ}Q{\circ}P = P \ \wedge \ Q{\circ}P{\circ}Q = Q \ .$$

Suppose $P \subseteq \mathbb{I} \wedge P \preceq Q$. We must verify (15). The first conjunct is verified as follows:

$$
\begin{array}{cl}
 & P \subseteq Q{\scriptstyle\asymp} \\
= & \{ \quad \text{assumption: } P \subseteq \mathbb{I} \text{ ; so } P = P{\scriptstyle\asymp} \quad \} \\
 & P{\scriptstyle\asymp} \subseteq Q{\scriptstyle\asymp} \\
\Leftarrow & \{ \quad \text{monotonicity} \quad \} \\
 & P \subseteq Q \\
\Leftarrow & \{ \quad (7) \quad \} \\
 & P \preceq Q \ .
\end{array}
$$

The second and third conjuncts both follow directly from the assumption $P \preceq Q$ by lemma 8.

For the converse implication, we have:

$$
\begin{array}{cl}
 & P \text{ is an index of } Q \\
\Rightarrow & \{ \quad \text{definition 2(a) and lemma 13 (both with } J,P := P,Q \text{)} \quad \} \\
 & P \subseteq Q{\scriptstyle\asymp} \ \wedge \ P \preceq Q \\
\Rightarrow & \{ \quad \text{domains (specifically } [\, P{\scriptstyle\asymp} \subseteq \mathbb{I} \,] \text{ with } P := Q \text{) and transitivity of } \subseteq \quad \} \\
 & P \subseteq \mathbb{I} \ \wedge \ P \preceq Q \ .
\end{array}
$$

□

Case analysis is a commonly used reasoning strategy but it is something we want to avoid whenever possible. Case analysis occurs when complements are used: for example, when using the law of the excluded middle. Using the cone rule also leads to case analysis: on when a relation is empty or non-empty. In section 6, a case analysis is unavoidable when interpreting our characterisation of maximality in terms of concrete relation: our theorem has the interpretation that a concrete relation is maximal with respect to the thins ordering iff it is empty or it is an equivalence relation. But we want to avoid such a case analysis in our formal calculations. Lemma 17 is crucial to our doing so. First, we need a general lemma.

Lemma 16 Suppose P is a per. Then

$$P{\scriptstyle\asymp} \ = \ (P{\circ}\mathbb{T}{\circ}P){\scriptstyle\asymp} \ .$$

(Note that, in general, $P{\circ}\mathbb{T}{\circ}P$ is a per if P is a per; the easy proof is left to the reader.)

Proof

$$\begin{array}{ll}
 & P_{\asymp} \\
= & \{ \quad P \text{ is a per, so } P = P\circ P \text{ (applied twice)} \quad \} \\
 & (P\circ P\circ P)_{\asymp} \\
\subseteq & \{ \quad P \subseteq \mathbb{T} \text{ ; monotonicity} \quad \} \\
 & (P\circ \mathbb{T}\circ P)_{\asymp} \\
\subseteq & \{ \quad \text{domains (specifically } [\,(R\circ S)_{<} \subseteq R_{<}\,] \text{ with } R,S := P\,,\mathbb{T}\circ P\,) \quad \} \\
 & P_{\asymp} \quad .
\end{array}$$

That is, by anti-symmetry, $P_{\asymp} = (P\circ\mathbb{T}\circ P)_{\asymp}$.

□

We have included the proof of lemma 16 because none of our earlier publications document the property. In what follows, we frequently use well-documented properties of coreflexives and domains. Typical of the sort of properties we use is that, for all pers P and coreflexives J ,

$$\mathbb{T}\circ P\circ J\circ P\circ \mathbb{T} \;=\; \mathbb{T}\circ J\circ P_{\asymp}\circ \mathbb{T} \quad .$$

The reader should be able to easily prove this property using the fact that coreflexives commute (i.e. for all coreflexives p and q , $p\circ q = q\circ p$) and are idempotents of composition (i.e. for all coreflexives p , $p\circ p = p$) and, for all pers P , $P_{\asymp}$ is coreflexive and $P_{\asymp}\circ\mathbb{T} = P\circ\mathbb{T}$ and $\mathbb{T}\circ P_{\asymp} = \mathbb{T}\circ P$. As explained earlier, and illustrated above, we often state the properties within the square "everywhere" brackets, with the convention that P ranges over pers and lower case letters (e.g. q) range over coreflexives.

For concrete relations, an index of a per is empty if and only if the per itself is empty. In the absence of the cone rule, a different idiom is needed to express such properties. This is the function of lemma 17.

Lemma 17 Suppose P is a per and J is an index of P . Then

$$\mathbb{T}\circ P\circ \mathbb{T} \;=\; \mathbb{T}\circ J\circ \mathbb{T} \quad .$$

Proof The proof is by mutual inclusion.

$$\begin{array}{ll}
 & \mathbb{T}\circ J\circ \mathbb{T} \\
\subseteq & \{ \quad J \text{ is an index of } P\text{, definition 2(a) and } [\; \mathbb{T}\circ P \;=\; \mathbb{T}\circ P_{\asymp} \;] \quad \} \\
 & \mathbb{T}\circ P\circ \mathbb{T} \\
= & \{ \quad J \text{ is an index of } P\text{, definition 2(c)} \quad \} \\
 & \mathbb{T}\circ P\circ J\circ P\circ \mathbb{T} \\
\subseteq & \{ \quad [\,U \subseteq \mathbb{T}\,] \text{ with } U := \mathbb{T}\circ P \text{ and } U := P\circ\mathbb{T} \quad \} \\
 & \mathbb{T}\circ J\circ \mathbb{T} \quad .
\end{array}$$

□

Lemma 18 Suppose P is a per, q is coreflexive and J is an index of $P\cup q$. Then $J\circ P_{\asymp}$ is an index of P and $q \subseteq P_{\asymp}\cup J$.

Proof The lemma implicitly assumes that $P\cup q$ is a per. This is easily verified.

Suppose J is an index of $P\cup q$. We verify the three defining properties of an index, 2(a), (b) and (c), with $J{,}P := J\circ P_{\asymp}\,,\,P$ assuming these properties with $J{,}P := J\,,\,P\cup q$. The first, $J\circ P_{\asymp} \subseteq P_{\asymp}$, is immediate from the fact that J is coreflexive. For the second, we have:

$J \circ P{<} \circ P \circ J \circ P{<}$

$=$ { J and $P{<}$ are coreflexive }

$J \circ P{<} \circ P \circ P{<} \circ J$

$=$ { $P{<}$ and q are coreflexive, and $P{<} \subseteq P$, so
$P{<} \circ q \circ P{<} \;\subseteq\; P{<} \circ P{<} \;=\; P{<} \circ P{<} \circ P{<} \;\subseteq\; P{<} \circ P \circ P{<}$ }

$J \circ (P{<} \circ P \circ P{<} \;\cup\; P{<} \circ q \circ P{<}) \circ J$

$=$ { distributivity }

$J \circ P{<} \circ (P \cup q) \circ P{<} \circ J$

$=$ { J and $P{<}$ are coreflexive }

$P{<} \circ J \circ (P \cup q) \circ J \circ P{<}$

$=$ { J is an index of $P \cup q$: definition 2(b) with $P,J := P{\cup}q\,,J$ }

$P{<} \circ J \circ P{<}$

$=$ { J and $P{<}$ are coreflexive }

$J \circ P{<}$.

Third,

true

$=$ { J is an index of $P \cup q$: definition 2(c) with $P,J := P{\cup}q\,,J$ }

$P{\cup}q \;=\; (P{\cup}q) \circ J \circ (P{\cup}q)$

$\Rightarrow$ { Leibniz }

$P \circ (P{\cup}q) \circ P \;=\; P \circ (P{\cup}q) \circ J \circ (P{\cup}q) \circ P$

$=$ { distributivity, $P{\circ}P = P$ }

$P \;\cup\; P{\circ}q{\circ}P \;=\; (P \cup P{\circ}q) \circ J \circ (P \cup q{\circ}P)$

$=$ { $q \subseteq \mathbb{I}$ and $P{\circ}P = P$, so $P{\circ}q{\circ}P \subseteq P$, $P{\circ}q \subseteq P$ and $q{\circ}P \subseteq P$ }

$P = P{\circ}J{\circ}P$

$=$ { $P{<} \circ P = P$ }

$P \;=\; P \circ J \circ P{<} \circ P$.

This completes the proof of the claim that $J \circ P{<}$ is an index of P. Finally,

q

$\subseteq$ { q is coreflexive, so $q = q{<}$; monotonicity }

$(P \cup q){<}$

$=$ { assumption: J is an index of $P \cup q$, definition 2(c) }

$((P \cup q) \circ J \circ (P \cup q)){<}$

$=$ { distributivity }

$(P{\circ}J{\circ}P \;\cup\; P{\circ}J{\circ}q \;\cup\; q{\circ}J{\circ}P \;\cup\; q{\circ}J{\circ}q){<}$

$\subseteq$ { $J \subseteq \mathbb{I}$ and $q \subseteq \mathbb{I}$, $P{\circ}P \subseteq P$ and monotonicity }

$(P \cup J){<}$

$=$ { distributivity; J is coreflexive, so $J{<} = J$ }

$P{<} \cup J$.

□

The remaining lemmas in this section are not needed elsewhere; they are included in order to give further insight into the nature of the thins ordering on pers.

Central to the notion of the thins relation is that an index of a per is found by successively "thinning" the relation. More precisely, an index of a per is a "thinning" of the per and being an index of a per is invariant under the process of "thinning" the relation. The first of these two properties is lemma 13; the second is formulated in lemma 19.

Lemma 19 For all pers P and Q, and coreflexive J,

$$J \text{ is an index of } Q \;\Leftarrow\; J \text{ is an index of } P \wedge P \preceq Q \;\;.$$

Proof We apply theorem 14:

$$\begin{array}{ll}
 & J \text{ is an index of } Q \\
= & \{ \quad \text{theorem 14 with } P,Q := J,Q \quad \} \\
 & J \subseteq \mathbb{I} \wedge J \preceq Q \\
\Leftarrow & \{ \quad \text{theorem 9 (in particular } \preceq \text{ is transitive)} \quad \} \\
 & J \subseteq \mathbb{I} \wedge J \preceq P \wedge P \preceq Q \\
= & \{ \quad \text{theorem 14 with } P,Q := J,P \quad \} \\
 & J \text{ is an index of } P \wedge P \preceq Q \;\;.
\end{array}$$

□

Our earlier informal interpretation of the first conjunct in the definition of thins is reinforced by the following simple lemma. Specifically, if J is an index of P, $J{\circ}P$ is the functional that maps a point a of P to the point in J that represents the equivalence class containing a. If $P \preceq Q$ then, by lemma 19, J is an index of Q. So, $J{\circ}Q$ is the functional that maps a point b of Q to the point in J that represents the equivalence class containing b. The lemma states that the two functionals agree on points common to both P and Q.

Lemma 20 Suppose J is an index of per P. Then, for all pers Q,

$$J{\circ}P \;=\; J \circ Q \circ P_{\succ\!\prec} \quad\Leftarrow\quad P \preceq Q \;\;.$$

Proof

$$\begin{array}{ll}
 & J{\circ}P \\
= & \{ \quad \text{assumption: } P \preceq Q \text{, definition 5} \quad \} \\
 & J \circ P_{\succ\!\prec} \circ Q \circ P_{\succ\!\prec} \\
= & \{ \quad J \text{ is an index of } P \text{, so, by definition 2(a), } J \circ P_{\succ\!\prec} = J \quad \} \\
 & J \circ Q \circ P_{\succ\!\prec} \;\;.
\end{array}$$

□

The final lemma in this section gives further insight into the relation between the thins relation and indexes.

Lemma 21 Suppose P and Q have a common index and $P \subseteq Q$. Then $P \preceq Q$.

Proof Suppose J is an index of both P and Q, and $P \subseteq Q$. The definition of the thins relation demands that we prove two properties. First,

$$
\begin{array}{ll}
 & Q \\
= & \{ \quad Q \text{ is a per, } [\, P = P \circ P \,] \text{ with } P := Q \quad \} \\
 & Q \circ Q \circ Q \\
\supseteq & \{ \quad \text{assumption: } P \subseteq Q \text{, monotonicity} \quad \} \\
 & Q \circ P \circ Q \\
\supseteq & \{ \quad P \text{ is a per, so } P_{\preceq} \subseteq P \quad \} \\
 & Q \circ P_{\preceq} \circ Q \\
\supseteq & \{ \quad \text{assumption: } J \text{ is an index of } P \text{, definition 2(a)} \quad \} \\
 & Q \circ J \circ Q \\
= & \{ \quad \text{assumption: } J \text{ is an index of } Q \text{, definition 2(c) (with } P := Q \text{)} \quad \} \\
 & Q \ .
\end{array}
$$

We conclude, by anti-symmetry, that $Q = Q \circ P_{\preceq} \circ Q$. Now for the second property,

$$
\begin{array}{ll}
 & P_{\preceq} \circ Q \circ P_{\preceq} \\
\supseteq & \{ \quad \text{assumption: } P \subseteq Q \text{, monotonicity} \quad \} \\
 & P_{\preceq} \circ P \circ P_{\preceq} \\
= & \{ \quad \text{domains} \quad \} \\
 & P \\
= & \{ \quad \text{assumption: } J \text{ is an index of } P \text{, definition 2(c)} \quad \} \\
 & P \circ J \circ P \\
= & \{ \quad \text{assumption: } J \text{ is an index of } Q \text{, definition 2(b) (with } P := Q \text{)} \quad \} \\
 & P \circ J \circ Q \circ J \circ P \\
= & \{ \quad Q \text{ is a per, } [\, P = P \circ P \,] \text{ with } P := Q \quad \} \\
 & P \circ J \circ Q \circ Q \circ Q \circ J \circ P \\
\supseteq & \{ \quad \text{assumption: } P \subseteq Q \text{, monotonicity} \quad \} \\
 & P \circ J \circ P \circ Q \circ P \circ J \circ P \\
= & \{ \quad \text{assumption: } J \text{ is an index of } P \text{, definition 2(c)} \quad \} \\
 & P \circ Q \circ P \\
\supseteq & \{ \quad P \text{ is a per, } [\, P_{\preceq} \subseteq P \,] \quad \} \\
 & P_{\preceq} \circ Q \circ P_{\preceq} \ .
\end{array}
$$

We conclude by anti-symmetry that $P_{\preceq} \circ Q \circ P_{\preceq} = P$. Combining the two calculations, we have shown that $P \preceq Q$. (See definition 5.)
□

5 Minimal Pers

Our goal in this section is to characterise the pers that are minimal with respect to the thins ordering on pers. Section 6 is about characterising the pers that are maximal. The notions of minimality and maximality with respect to an ordering relation are well known. For completeness the definition is given below.

Definition 22 Suppose $\sqsubseteq$ is a partial ordering on some set X. With x and y ranging over elements of X, we say that y is *minimal* with respect to the ordering iff

$$\langle \forall x \,:\, x \sqsubseteq y \,:\, x = y \rangle$$

and we say that x is *maximal* with respect to the ordering iff

$$\langle \forall y : x \sqsubseteq y : x = y \rangle \ .$$

□

We apply definition 22 in this section and in section 6 to the thins ordering on pers; in section 7 we apply the definition to the (yet-to-be-introduced) thins ordering on arbitrary relations.

A straightforward observation is that coreflexives are minimal:

Lemma 23

$$\langle \forall Q :: \mathsf{minimal}.Q \Leftarrow Q \subseteq \mathbb{I} \rangle \ .$$

Proof Suppose $Q \subseteq \mathbb{I}$ and $P \preceq Q$. We prove that $P = Q$.

$$
\begin{array}{ll}
 & P = Q \\
= & \quad \{ \quad \text{anti-symmetry} \quad \} \\
 & P \subseteq Q \ \wedge \ Q \subseteq P \\
= & \quad \{ \quad \text{assumption: } P \preceq Q \text{ ; hence, by (7), } P \subseteq Q \quad \} \\
 & Q \subseteq P \\
= & \quad \{ \quad \text{assumption: } P \preceq Q \text{ ; hence by definition 5, } Q = Q \circ P \!\asymp\! \circ Q \quad \} \\
 & Q \circ P \!\asymp\! \circ Q \subseteq P \\
\Leftarrow & \quad \{ \quad \text{assumption: } Q \subseteq \mathbb{I} \text{ ; monotonicity} \quad \} \\
 & P \!\asymp\! \subseteq P \\
= & \quad \{ \quad P \text{ is a per} \quad \} \\
 & \mathsf{true} \ .
\end{array}
$$

□

Lemma 23 suggests that we explore the circumstances in which all minimal elements are coreflexives. We show that this is equivalent to the axiom of choice introduced in [BV22, BV23b].

Theorem 24 The axiom of choice, axiom 3, is equivalent to

(25) $\langle \forall P :: \mathsf{minimal}.P \equiv P \subseteq \mathbb{I} \rangle \ \wedge \ \langle \forall Q :: \langle \exists P : \mathsf{minimal}.P : P \preceq Q \rangle \rangle$

Proof The proof is by mutual implication. First, assume (25). Suppose Q is an arbitrary per. We prove that Q has an index.

By assumption, there exists a per P such that $P \subseteq \mathbb{I}$ and $P \preceq Q$. Theorem 14 proves that P is an index of Q.

Now assume the axiom of choice. We must prove (25). We begin with the property

(26) $\langle \forall P :: \mathsf{minimal}.P \equiv P \subseteq \mathbb{I} \rangle \ .$

By lemma 23, it suffices to prove the implication. Suppose P is minimal. By the axiom of choice, P has an index, J say. Then

$$
\begin{array}{ll}
 & \mathsf{true} \\
= & \{ \quad \mathsf{J} \text{ is an index of } \mathsf{P}\text{; lemma 13} \quad \} \\
 & \mathsf{J} \preceq \mathsf{P} \\
\Rightarrow & \{ \quad \mathsf{P} \text{ is minimal; definition 22} \quad \} \\
 & \mathsf{P} = \mathsf{J} \\
\Rightarrow & \{ \quad \mathsf{J} \text{ is an index of } \mathsf{P}\text{, so, by definition 2(a), } \mathsf{J} \subseteq \mathsf{P}{\asymp}\text{; } [\, \mathsf{P}{\asymp} \subseteq \mathbb{I} \,] \quad \} \\
 & \mathsf{P} \subseteq \mathbb{I} \ .
\end{array}
$$

We have thus proved (26). Now we consider the property

(27) $\langle \forall \mathsf{Q} :: \langle \exists \mathsf{P} : \mathsf{minimal.P} : \mathsf{P} \preceq \mathsf{Q} \rangle \rangle$.

This is established by choosing, for given Q, an index of Q. Indeed, if P is an index of Q, then, by lemma 13, $\mathsf{P} \preceq \mathsf{Q}$ and, by (26), it is minimal.
□

Example 28 As observed in example 4, the algebras of example 1 with at least three elements do not satisfy our axiom of choice. Consequently, they do not satisfy the minimality property of theorem 24: in both cases, $\mathbb{T}$ is minimal but not coreflexive. (See example 6.)
□

6 Maximal Pers

In this section we formulate a necessary and sufficient condition guaranteeing that a given per is maximal with respect to the thins ordering. See theorem 52.

It is relatively straightforward to show that $\perp\!\!\!\perp$ is maximal and all equivalence relations are maximal. This suggests the conjecture that these are the only maximal elements. Rather than formulate a proof that involves a case analysis, we prove a more general property. Specifically, we prove that a per P is maximal iff $\mathbb{I} \cap \mathbb{T} \circ \mathsf{P} \circ \mathbb{T} \subseteq \mathsf{P}$. If the cone rule holds, $\mathbb{I} \cap \mathbb{T} \circ \mathsf{P} \circ \mathbb{T} \subseteq \mathsf{P}$ is equivalent to $\mathbb{I} \subseteq \mathsf{P} \vee \mathsf{P} = \perp\!\!\!\perp$. The additional generality comes from instances of relation algebra where the cone rule does not hold.

6.1 Sufficient Condition for Maximality

First, we prove the "if" statement.

Lemma 29 Per P is maximal if $\mathbb{I} \cap \mathbb{T} \circ \mathsf{P} \circ \mathbb{T} \subseteq \mathsf{P}$.

Proof Suppose $\mathbb{I} \cap \mathbb{T} \circ \mathsf{P} \circ \mathbb{T} \subseteq \mathsf{P}$ and $\mathsf{P} \preceq \mathsf{Q}$. To show that P is maximal we must show that $\mathsf{P} = \mathsf{Q}$. We first show that $\mathsf{P}{\asymp} = \mathsf{Q}{\asymp}$.

$$
\begin{array}{ll}
 & \mathsf{Q}{\asymp} \\
= & \{ \quad \mathsf{Q} \text{ is a per} \quad \} \\
 & \mathbb{I} \cap \mathsf{Q} \\
= & \{ \quad \mathsf{P} \preceq \mathsf{Q}\text{, lemma 8} \quad \} \\
 & \mathbb{I} \cap \mathsf{Q} \circ \mathsf{P} \circ \mathsf{Q}
\end{array}
$$

$\subseteq$ { $Q \subseteq \mathbb{T}$ and monotonicity }
$\mathbb{I} \cap \mathbb{T}\circ P\circ \mathbb{T}$
$=$ { domains }
$(\mathbb{I} \cap \mathbb{T}\circ P\circ \mathbb{T})_{\asymp}$
$\subseteq$ { assumption: $\mathbb{I}\cap \mathbb{T}\circ P\circ \mathbb{T} \subseteq P$ and monotonicity }
$P_{\asymp}$
$\subseteq$ { assumption: $P \preceq Q$, (7) and monotonicity }
$Q_{\asymp}$

Thus $P_{\asymp} = Q_{\asymp}$ follows by anti-symmetry. Now we show that $P = Q$:

P
$=$ { assumption: $P \preceq Q$ }
$P_{\asymp} \circ Q \circ P_{\asymp}$
$=$ { $P_{\asymp} = Q_{\asymp}$ (just proved) }
$Q_{\asymp} \circ Q \circ Q_{\asymp}$
$=$ { domains }
Q .

□

6.2 Necessary Condition for Maximality

We now turn to the converse of lemma 29. The key fact is theorem 30.

Theorem 30 Suppose P is an arbitrary per. Then, assuming the axiom of choice, there is a per Q such that $\mathbb{I}\cap \mathbb{T}\circ Q\circ \mathbb{T} \subseteq Q$ and $P \preceq Q$.
□

The remainder of this section is about proving this theorem. Throughout we assume that P is an arbitrary per and q , J , R and Q are defined by (31), (32), (33) and (34):

(31) $q = \mathbb{I} \cap \mathbb{T}\circ P\circ \mathbb{T}$,

(32) J is an index of $P\circ \mathbb{T}\circ P \cup q$,

(33) $R = J\circ \mathbb{T}\circ J$,

(34) $Q = P \cup R \cup P\circ R \cup R\circ P$.

The heuristics that lead to these definitions are as follows. These heuristics are based on the interpretation of the elements of our algebra as concrete relations and, as such, contain implicitly several properties on which we do not rely. Great care must therefore be taken with such interpretations.

Suppose P is an arbitrary per. To prove theorem 30 we have to construct a per Q with two properties, the first of which is $\mathbb{I}\cap \mathbb{T}\circ Q\circ \mathbb{T} \subseteq Q$. For concrete relations, the interpretation of this property is that Q is either the empty relation, or that Q is an equivalence relation (a per with domain $\mathbb{I}$). For concrete relations, this leads to a case analysis on whether or not P is the empty relation: in the case that P *is* the empty relation, Q is defined to also be the empty relation, and in the case that P is *not* the empty relation, it is necessary to define Q to be an equivalence relation. In what follows, we assume that P is non-empty.

The second requirement on Q is $P \preceq Q$. In order to achieve this additional goal, we aim to define Q to be an equivalence relation that extends P by adding all points not in the domain of P to one of the equivalence classes of P. This goal involves several elements. Extending P means defining Q as $P \cup U$ for some U; adding points to one of the equivalence classes of P entails choosing one such class; finally, ensuring that Q is an equivalence relation means guaranteeing that the domain of Q is $\mathbb{I}$ and Q is both symmetric and transitive.

In general, $P \cup U$ is not a per, even if U is a per. The required transitivity is readily satisfied if Q is defined as $(P \cup R)^{+}$ for some R; moreover the required symmetry is also satisfied if we choose for R a symmetric relation — in particular, if R is itself a per[3]. Importantly, if P and R are both pers, $(P \cup R)^{+}$ is the smallest per that contains both P and R.

The goal becomes to define Q to be $(P \cup R)^{+}$ where R is a per so defined that its domain includes one of the equivalence classes of P together with all points that are not in the domain of P. Choosing one of the eqivalence classes of P is achieved by using our axiom of choice.

At this point, we need to anticipate the fact that we do not want to assume the cone rule. Interpreted as a concrete relation, $\mathbb{T} \circ P \circ \mathbb{T}$ is either the empty relation or the universal relation, depending on whether P is empty or non-empty. The interpretation of $\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T}$ is thus either the empty set of points or the set of all points. This interpretation relies on the cone rule, which we do not wish to exploit. We therefore introduce the abbreviation q for $\mathbb{I} \cap \mathbb{T} \circ P \circ \mathbb{T}$. In the current context — P is a non-empty, concrete relation— $q = \mathbb{I}$. References to q below, rather than $\mathbb{I}$, anticipate the more general property that we actually prove.

Interpreted as a concrete relation, the relation $P \circ \mathbb{T} \circ P$ is a per with a single equivalence class that contains all the points in the domain of P. By choosing an index of $P \circ \mathbb{T} \circ P$, we effectively choose one point in the domain of P; an index J of $P \circ \mathbb{T} \circ P \cup q$ is thus interpreted as a set of points consisting of one point in the domain of P together with all the points in q that are not in the domain of P. The interpretation of $J \circ P_{\approx}$ is the point in the domain of P chosen by J; the formal properties of $J \circ P_{\approx}$ play a central role in the proof.

The interpretation of $J \circ \mathbb{T} \circ J$ is a per with exactly one equivalence class: the class containing all the points equivalent in P to the point $J \circ P_{\approx}$ together with all the points in q that are not in the domain of P. Thus $J \circ \mathbb{T} \circ J$ is the per R needed to achieve our objective.

There is one more problem to be resolved. We want to define Q to be the transitive closure of $P \cup R$: the least transitive relation that includes both P and R (and hence, since R is a per, the least per that includes both P and R). But, for arbitrary concrete relation U, its transitive closure U^{+} is computed by a possibly non-terminating process: beginning with U continually add to U successive powers of U (thus computing U, $U \cup U \circ U$, $U \cup U \circ U \cup U \circ U \circ U$, and so on). Fortunately, in the case of $P \cup R$ where R is defined by (33),

[3]For arbitrary (homogeneous) relation U, U^{+} denotes the transitive closure of U — the smallest relation that includes U and is transitive. For the purposes of the current informal account, we assume familiarity with properties of U^{+}. For example, we assume the reader is familiar with the property that, for all U, $(U^{+})^{\cup} = (U^{\cup})^{+}$. The formal calculations do not make any such assumption.

this process is terminating; indeed, it terminates after the first iteration. The definition of Q above anticipates this fact. See section 6.2.2.

Note that, in the above informal account, we have been obliged to refer to "points" and to points "not" having a certain property. It is important to note that our calculations make no assumptions about the existence of points or the existence of complements. The above informal account is applicable only to concrete relations.

Let us now proceed with the proof of theorem 30 where Q is defined by (34). In section 6.2.1 we establish a number of properties of the index J; that Q is a per is proved in section 6.2.2; finally, section 6.2.3 is where theorem 30 is proved.

6.2.1 Properties of the Index J

The focus in this subsection is on the properties of the index J. The definition of J assumes that $P{\circ}\mathbb{T}{\circ}P \cup q$ is a per. This is a straightforward consequence of the fact that, in general, $P{\cup}q$ is a per if P is a per and q is coreflexive, and $P{\circ}\mathbb{T}{\circ}P$ is a per if P is a per. The details are left to the reader.

We now proceed to exploit definition (32). A key property is lemma 37; other lemmas are used either to establish these two lemmas or in later calculations.

Lemma 35

$$(P{\circ}\mathbb{T}{\circ}P \cup q)_{\ltimes} = q \quad .$$

Proof

$$\begin{array}{ll}
 & (P{\circ}\mathbb{T}{\circ}P \cup q)_{\ltimes} \\
= & \quad \{ \quad \text{distributivity property of domain operator} \quad \} \\
 & (P{\circ}\mathbb{T}{\circ}P)_{\ltimes} \cup q_{\ltimes} \\
= & \quad \{ \quad \text{lemma 16 and } q \text{ is coreflexive} \quad \} \\
 & P_{\ltimes} \cup q \\
= & \quad \{ \quad \text{definition of } q: (31) \quad \} \\
 & P_{\ltimes} \cup (\mathbb{I} \cap \mathbb{T}{\circ}P{\circ}\mathbb{T}) \\
= & \quad \{ \quad [\ P_{\ltimes} = \mathbb{I}{\cap}P \subseteq \mathbb{I} \cap \mathbb{T}{\circ}P{\circ}\mathbb{T}\ ] \quad \} \\
 & \mathbb{I} \cap \mathbb{T}{\circ}P{\circ}\mathbb{T} \\
= & \quad \{ \quad \text{definition of } q: (31) \quad \} \\
 & q \quad .
\end{array}$$

□

Now J is defined to be an index of $P{\circ}\mathbb{T}{\circ}P \cup q$. So, applying definition 2(a), an immediate corollary of lemma 35 is that

$$(36) \qquad J \subseteq q \quad .$$

The interpretation of $J \circ P_{\ltimes}$ in terms of concrete relations is the point chosen by the index J that indexes $P{\circ}\mathbb{T}{\circ}P$; the other points in J are the points in q that are not in $P_{\ltimes}$. Aspects of these informal interpretations that are relevant to our calculations are expressed by lemma 37.

Lemma 37 $\;J \circ P{\asymp}$ is an index of $P{\circ}\mathbb{T}{\circ}P$. Hence

(38) $\quad J \circ P{\asymp} \;=\; J \circ P{\asymp} \circ \mathbb{T} \circ P{\asymp} \circ J$,

(39) $\quad P{\circ}\mathbb{T}{\circ}P \;=\; P \circ \mathbb{T} \circ J \circ P{\asymp} \circ \mathbb{T} \circ P$,

(40) $\quad \mathbb{T} \circ P{\asymp} \circ \mathbb{T} \;=\; \mathbb{T} \circ J \circ P{\asymp} \circ \mathbb{T}$, and

(41) $\quad q \subseteq P{\asymp} \cup J$.

Proof

$$
\begin{array}{ll}
 & \text{true} \\
= & \{ \quad \text{lemma 18 with } J,P := J\,,\, P{\circ}\mathbb{T}{\circ}P \cup q \quad \} \\
 & J \circ (P{\circ}\mathbb{T}{\circ}P){\asymp} \text{ is an index of } P{\circ}\mathbb{T}{\circ}P \\
= & \{ \quad \text{lemma 16} \quad \} \\
 & J \circ P{\asymp} \text{ is an index of } P{\circ}\mathbb{T}{\circ}P \;\; .
\end{array}
$$

We now derive properties (38), (39), (40) and (41). First (38):

$$
\begin{array}{ll}
 & J \circ P{\asymp} \\
= & \{ \quad \text{definition 2(b) with } J,P := J \circ P{\asymp}\,,\, P{\circ}\mathbb{T}{\circ}P \quad \} \\
 & J \circ P{\asymp} \circ P \circ \mathbb{T} \circ P \circ J \circ P{\asymp} \\
= & \{ \quad [\, P{\asymp} \circ P = P = P \circ P{\asymp} \,],\; J \text{ and } P{\asymp} \text{ are coreflexive, so } J \circ P{\asymp} = P{\asymp} \circ J \quad \} \\
 & J \circ P{\asymp} \circ \mathbb{T} \circ P{\asymp} \circ J \;\; .
\end{array}
$$

Property (39) follows:

$$
\begin{array}{ll}
 & P{\circ}\mathbb{T}{\circ}P \\
= & \{ \quad \text{definition 2(c) with } J,P := J \circ P{\asymp}\,,\, P{\circ}\mathbb{T}{\circ}P \quad \} \\
 & P \circ \mathbb{T} \circ P \circ J \circ P{\asymp} \circ P \circ \mathbb{T} \circ P \\
= & \{ \quad [\, \mathbb{T}{\circ}P = \mathbb{T} \circ P{\asymp} \,],\; [\, P{\asymp} \circ J = J \circ P{\asymp} \,],\; [\, P{\asymp} \circ P = P \,] \quad \} \\
 & P \circ \mathbb{T} \circ J \circ P{\asymp} \circ \mathbb{T} \circ P \;\; .
\end{array}
$$

Next (40):

$$
\begin{array}{ll}
 & \mathbb{T} \circ J \circ P{\asymp} \circ \mathbb{T} \\
= & \{ \quad \text{lemma 17 with } J,P := J \circ P{\asymp}\,,\, P{\circ}\mathbb{T}{\circ}P \quad \} \\
 & \mathbb{T}{\circ}P{\circ}\mathbb{T}{\circ}P{\circ}\mathbb{T} \\
= & \{ \quad [\, \mathbb{T}{\circ}P = \mathbb{T} \circ P{\asymp} \,] \text{ with } P := P{\circ}\mathbb{T}{\circ}P\,,\, \text{lemma 16} \quad \} \\
 & \mathbb{T} \circ P{\asymp} \circ \mathbb{T} \;\; .
\end{array}
$$

Finally (41):

$$
\begin{array}{ll}
 & q \subseteq P{\asymp} \cup J \\
= & \{ \quad \text{lemma 16} \quad \} \\
 & q \subseteq (P{\circ}\mathbb{T}{\circ}P){\asymp} \cup J \\
= & \{ \quad \text{assumption: } J \text{ is an index of } P{\circ}\mathbb{T}{\circ}P \cup q\,,\, \text{lemma 18 with } P := P{\circ}\mathbb{T}{\circ}P \quad \} \\
 & \text{true} \;\; .
\end{array}
$$

□

If p is coreflexive and non-empty, the (concrete-relational) interpretation of $p\circ\mathbb{T}\circ p$ is a per with domain p that has exactly one equivalence class; the interpretation of (38) is thus that $J\circ P_{\asymp}$ is a single point that is in both J and the domain of P. That is, $J\circ P_{\asymp}$ is the point in the domain of P chosen by the index J. Property (41) is necessitated by our wish to avoid assuming the cone rule. For concrete relations, it is interpreted as the property that $P_{\asymp}\cup J$ is $\mathbb{I}$ if P is non-empty.

6.2.2 Q is a per

A basic requirement on Q is that it is a per, i.e. symmetric and transitive. The transitivity of Q is not obvious. The complication is that we do not wish to exploit the cone rule. Instead, we have to prove several lemmas that are trivial if the cone rule is assumed: lemmas 42, 43 and 44. Lemma 45 is also needed.

Lemma 42

$$J\circ\mathbb{T}\circ J \;=\; J\circ\mathbb{T}\circ P\circ\mathbb{T}\circ J \quad .$$

Proof The proof is by mutual inclusion.

$$\begin{array}{cl}
 & J\circ\mathbb{T}\circ P\circ\mathbb{T}\circ J \\
\subseteq & \{\quad [U\subseteq\mathbb{T}] \text{ with } U:=\mathbb{T}\circ P\circ\mathbb{T}\text{, monotonicity} \quad\} \\
 & J\circ\mathbb{T}\circ J \\
= & \{\quad (36),\ J \text{ and } q \text{ are coreflexive (so } J\circ q = J) \quad\} \\
 & J\circ q\circ\mathbb{T}\circ J \\
\subseteq & \{\quad \text{definition of } q:\ (31)\text{, monotonicity} \quad\} \\
 & J\circ\mathbb{T}\circ P\circ\mathbb{T}\circ\mathbb{T}\circ J \\
\subseteq & \{\quad [U\subseteq\mathbb{T}] \text{ with } U:=\mathbb{T}\circ\mathbb{T} \quad\} \\
 & J\circ\mathbb{T}\circ P\circ\mathbb{T}\circ J \quad .
\end{array}$$

□

Lemma 43

$$R\circ P\circ R \;=\; R \;=\; R\circ P_{\asymp}\circ R \quad .$$

Proof The proof is by mutual inclusion. Recalling the definition of R, (33), we have:

$$\begin{array}{cl}
 & J\circ\mathbb{T}\circ J\circ P\circ J\circ\mathbb{T}\circ J \\
\subseteq & \{\quad [U\subseteq\mathbb{T}] \text{ with } U:=\mathbb{T}\circ J\circ P\circ J\circ\mathbb{T} \quad\} \\
 & J\circ\mathbb{T}\circ J \\
= & \{\quad \text{lemma 42} \quad\} \\
 & J\circ\mathbb{T}\circ P\circ\mathbb{T}\circ J \\
= & \{\quad [\ \mathbb{T}\circ P = \mathbb{T}\circ P_{\asymp}\ ]\text{, (40)} \quad\} \\
 & J\circ\mathbb{T}\circ J\circ P_{\asymp}\circ\mathbb{T}\circ J
\end{array}$$

$=\quad\{\quad J$ and $P_{\asymp}$ are coreflexive, so $J\circ P_{\asymp} = J\circ J\circ P_{\asymp} = J\circ P_{\asymp}\circ J \quad\}$

$J\circ\mathbb{T}\circ J\circ P_{\asymp}\circ J\circ\mathbb{T}\circ J$

$\subseteq\quad\{\quad P_{\asymp}\subseteq P \quad\}$

$J\circ\mathbb{T}\circ J\circ P\circ J\circ\mathbb{T}\circ J$

We conclude, by mutual inclusion, that

$$J\circ\mathbb{T}\circ J\circ P\circ J\circ\mathbb{T}\circ J \;=\; J\circ\mathbb{T}\circ J \;=\; J\circ\mathbb{T}\circ J\circ P_{\asymp}\circ J\circ\mathbb{T}\circ J \quad .$$

The lemma follows by instantiating the definition of R.

□

Lemma 44

$$\mathbb{T}\circ P\circ\mathbb{T} \;=\; \mathbb{T}\circ Q\circ\mathbb{T} \quad .$$

Proof

$\mathbb{T}\circ Q\circ\mathbb{T}$

$=\quad\{\quad$ definition: (34) and distributivity $\quad\}$

$\mathbb{T}\circ P\circ\mathbb{T} \;\cup\; \mathbb{T}\circ R\circ\mathbb{T} \;\cup\; \mathbb{T}\circ P\circ R\circ\mathbb{T} \;\cup\; \mathbb{T}\circ R\circ P\circ\mathbb{T}$

$=\quad\{\quad [\,\mathbb{T}\circ R\subseteq\mathbb{T}\,]$ and $[\,R\circ\mathbb{T}\subseteq\mathbb{T}\,]$,

so $[\;\mathbb{T}\circ P\circ R\circ\mathbb{T} \;\cup\; \mathbb{T}\circ R\circ P\circ\mathbb{T} \;\subseteq\; \mathbb{T}\circ P\circ\mathbb{T}\;] \quad\}$

$\mathbb{T}\circ P\circ\mathbb{T} \;\cup\; \mathbb{T}\circ R\circ\mathbb{T}$

$=\quad\{\quad$ lemma 43 $\quad\}$

$\mathbb{T}\circ P\circ\mathbb{T} \;\cup\; \mathbb{T}\circ R\circ P\circ R\circ\mathbb{T}$

$=\quad\{\quad [\,\mathbb{T}\circ R\subseteq\mathbb{T}\,]$ and $[\,R\circ\mathbb{T}\subseteq\mathbb{T}\,]$, so $[\;\mathbb{T}\circ R\circ P\circ R\circ\mathbb{T}\subseteq\mathbb{T}\circ P\circ\mathbb{T}\;] \quad\}$

$\mathbb{T}\circ P\circ\mathbb{T} \quad .$

□

Lemma 45

$$P\circ R\circ P\subseteq P \quad .$$

Proof

$P\circ R\circ P$

$=\quad\{\quad$ definition: (33) $\quad\}$

$P\circ J\circ\mathbb{T}\circ J\circ P$

$=\quad\{\quad [\;P\circ P_{\asymp} = P = P_{\asymp}\circ P\;] \quad\}$

$P\circ P_{\asymp}\circ J\circ\mathbb{T}\circ J\circ P_{\asymp}\circ P$

$=\quad\{\quad J$ and $P_{\asymp}$ are coreflexive, so $P_{\asymp}\circ J = J\circ P_{\asymp}$, (38) $\quad\}$

$P\circ J\circ P_{\asymp}\circ P$

$\subseteq\quad\{\quad J$ and $P_{\asymp}$ are coreflexive, and monotonicity $\quad\}$

$P\circ P$

$=\quad\{\quad P$ is a per $\quad\}$

$P \quad .$

□

We now show more than just transitivity of Q: we show that Q is the transitive closure of $P \cup R$.

Lemma 46

(47) $\quad Q \;=\; (P\cup R)\circ(P\cup R) \;=\; (P\cup R)\circ P^{\asymp}\circ(P\cup R)$, and

(48) $\quad Q\circ(P\cup R) \;=\; Q$.

Proof Property (47) is easy to prove:

$$(P\cup R)\circ P^{\asymp}\circ(P\cup R)$$
$$= \quad \{ \quad \text{distributivity, } P\circ P^{\asymp} = P = P^{\asymp}\circ P \text{ and } P = P\circ P \quad \}$$
$$P \,\cup\, R\circ P \,\cup\, P\circ R \,\cup\, R\circ P^{\asymp}\circ R$$
$$= \quad \{ \quad \text{lemma 43} \quad \}$$
$$P \cup R\circ P \cup P\circ R \cup R$$
$$= \quad \{ \quad P \text{ and } R \text{ are pers, so } P = P\circ P \text{ and } R = R\circ R \quad \}$$
$$P\circ P \cup R\circ R \cup P\circ R \cup R\circ P$$
$$= \quad \{ \quad \text{distributivity} \quad \}$$
$$(P\cup R)\circ(P\cup R) \ .$$

Applying (34), the definition of Q (and using the symmetry of set union), this proves (47). Turning now to (48), we have:

$$Q\circ(P\cup R)$$
$$= \quad \{ \quad \text{definition: (34)} \quad \}$$
$$(P \cup R \cup P\circ R \cup R\circ P)\circ(P\cup R)$$
$$= \quad \{ \quad \text{distributivity and } P \text{ and } R \text{ are pers (so } P\circ P = P \text{ and } R\circ R = R\text{)} \quad \}$$
$$P \cup R\circ P \cup P\circ R \cup R \cup P\circ R\circ P \cup R\circ P\circ R$$
$$= \quad \{ \quad \text{lemmas 45 and 43} \quad \}$$
$$P \cup R\circ P \cup P\circ R \cup R$$
$$= \quad \{ \quad \text{definition: (34)} \quad \}$$
$$Q \ .$$

□

Corollary 49 $\quad Q$ is the least transitive relation that includes $P\cup R$. That is, Q is the transitive closure of $P\cup R$. Also, Q is symmetric and, hence, a per.

Proof First we show that Q is transitive:

$$Q\circ Q$$
$$= \quad \{ \quad (47) \quad \}$$
$$Q\circ(P\cup R)\circ(P\cup R)$$
$$= \quad \{ \quad \text{(48) (applied twice)} \quad \}$$
$$Q \ .$$

Now we show that it is least among all such relations: we have, for all X,

$$
\begin{array}{ll}
 & P \cup R \subseteq X \;\wedge\; X \circ X \subseteq X \\
\Rightarrow & \{ \quad \text{monotonicity and transitivity of } \subseteq \quad \} \\
 & (P \cup R) \circ (P \cup R) \subseteq X \\
= & \{ \quad \text{lemma 46} \quad \} \\
 & Q \subseteq X \;\; .
\end{array}
$$

That is, Q is the transitive closure of $P \cup R$.

Symmetry of Q is obvious from (47) (specifically, $Q = (P \cup R) \circ (P \cup R)$) and the fact that P and R are both pers.

□

Corollary 49 is not used directly in its entirety, only transitivity and symmetry being explicitly invoked. It is included in order to provide further justification for the definition of Q. Specifically, Q is the least per that includes both P and R. That it includes both P and R means that all the points in the index J are combined with the points in the equivalence class in P defined by the point in $P_{\gtrless}$ chosen by J; that it is least means that other equivalence classes of P are unaffected.

6.2.3 Proof of Theorem 30

We are now in a position to prove the two properties of Q required by theorem 30. See lemmas 50 and 51 below.

Lemma 50

$$P \preceq Q \;\; .$$

Proof Recalling definition 5, we have to prove two properties. First,

$$
\begin{array}{ll}
 & P_{\gtrless} \circ Q \circ P_{\gtrless} \\
= & \{ \quad (34) \quad \} \\
 & P_{\gtrless} \circ (P \cup R \cup P \circ R \cup R \circ P) \circ P_{\gtrless} \\
= & \{ \quad \text{distributivity and domains (specifically } [\; P_{\gtrless} \circ P = P = P \circ P_{\gtrless} \;]) \quad \} \\
 & P \;\cup\; P_{\gtrless} \circ R \circ P_{\gtrless} \;\cup\; P \circ R \circ P_{\gtrless} \;\cup\; P_{\gtrless} \circ R \circ P \\
= & \{ \quad P_{\gtrless} \subseteq P \text{ and lemma 45} \quad \} \\
 & P \;\; .
\end{array}
$$

Second,

$$
\begin{array}{ll}
 & Q \circ P_{\gtrless} \circ Q \\
= & \{ \quad \text{(48) and, hence, by symmetry, } (P \cup R) \circ Q \;=\; Q \quad \} \\
 & Q \circ (P \cup R) \circ P_{\gtrless} \circ (P \cup R) \circ Q \\
= & \{ \quad (47) \quad \} \\
 & Q \circ Q \circ Q \\
= & \{ \quad Q \text{ is a per (corollary 49), so } Q \circ Q \quad \} \\
 & Q \;\; .
\end{array}
$$

□

Lemma 51

$$\mathbb{I} \cap \mathbb{T}{\circ}Q{\circ}\mathbb{T} \;\subseteq\; Q \quad .$$

Proof

$$
\begin{array}{ll}
 & \mathbb{I} \cap \mathbb{T}{\circ}Q{\circ}\mathbb{T} \\
= & \{ \qquad \text{lemma 44} \quad \} \\
 & \mathbb{I} \cap \mathbb{T}{\circ}P{\circ}\mathbb{T} \\
= & \{ \qquad \text{definition: (31)} \quad \} \\
 & q \\
\subseteq & \{ \qquad \text{lemma 41} \quad \} \\
 & P{\gtrless} \cup J \\
= & \{ \qquad \text{definition of } R\text{: (33), lemma 16 with } P{:=}R \quad \} \\
 & P{\gtrless} \cup R{\gtrless} \\
\subseteq & \{ \qquad [\, P{\gtrless} \subseteq P \,] \text{ with } P{:=}P \text{ and } P{:=}R\text{; monotonicity} \quad \} \\
 & P \cup R \\
\subseteq & \{ \qquad \text{definition: (34), and } [\, U \subseteq U{\cup}V \,] \quad \} \\
 & Q \quad .
\end{array}
$$

□

Lemmas 50 and 51 conclude the proof of theorem 30.

Theorem 52 A per P is maximal with respect to the thins ordering iff $\mathbb{I}\cap\mathbb{T}{\circ}P{\circ}\mathbb{T} \subseteq P$.

Proof We have shown (lemma 29) that P is maximal if $\mathbb{I}\cap\mathbb{T}{\circ}P{\circ}\mathbb{T} \subseteq P$. Only-if follows from theorem 30. Specifically, suppose P is a per. By theorem 30, there is a per Q such that $\mathbb{I}\cap\mathbb{T}{\circ}Q{\circ}\mathbb{T} \subseteq Q$ and $P \preceq Q$. So, by definition of maximal, if P is maximal, $P = Q$. That is, by Leibniz's rule, if P is maximal, $\mathbb{I}\cap\mathbb{T}{\circ}P{\circ}\mathbb{T} \subseteq P$.

□

It is remarkable that the proof of theorem 52 does not rely in any way on saturation properties of the lattices of coreflexives or of relations in general. The proof is entirely "point-free". Equally remarkable is that nowhere do we use complements.

It is also worth emphasising that we have avoided the use of the cone rule and, in so doing, have avoided a case analysis in the statement of theorem 52. This means that the theorem is also applicable for non-unary relation algebras. For concrete relations (where the cone rule does apply), the interpretation of theorem 52 is that a per P is maximal iff it is empty or is an equivalence relation. (This is because, by applying the cone rule, the property $\mathbb{I}\cap\mathbb{T}{\circ}P{\circ}\mathbb{T} \subseteq P$ simplifies to $P = \perp\!\!\!\perp \vee I \subseteq P$.)

7 Extending thins to arbitrary relations

In this section, we extend the thins ordering to arbitrary relations. The section is concluded by theorem 64 which states that the minimal elements of the extended ordering are exactly the core relations introduced in [BV22, BV23b].

Recall that $R_{\prec}$ denotes the left per-domain of R and $R_{\succ}$ denotes its right per-domain.

Definition 53 (Thins) For arbitrary relations R and S of the same type, the relation $R \preceq S$ is defined by

$$R \preceq S \quad \equiv \quad R_{\prec} \preceq S_{\prec} \;\wedge\; R_{\succ} \preceq S_{\succ} \;\wedge\; R = R_{<} \circ S \circ R_{>} \quad .$$

□

The symbol " $\preceq$ " is overloaded in definition 53. If R and S have type $A \sim B$, the leftmost occurrence is a relation on relations of type $A \sim B$, the middle occurrence is a relation on pers of type A and the rightmost occurrence is a relation on pers of type B.

Lemma 54 The thins relation on arbitrary relations is an ordering relation.

Proof The thins relation on arbitrary relations is clearly reflexive. Transitivity is also easy to prove. Anti-symmetry is proven below.

$$\begin{array}{ll}
 & R \preceq S \;\wedge\; S \preceq R \\
\Rightarrow & \{ \quad \text{definition 53 with } R,S := R,S \text{ and } R,S := S,R \quad \} \\
 & R = R_{<} \circ S \circ R_{>} \;\wedge\; S = S_{<} \circ R \circ S_{>} \\
\Rightarrow & \{ \quad \text{domains} \quad \} \\
 & (R_{<} \subseteq S_{<} \;\wedge\; R_{>} \subseteq S_{>}) \wedge (S_{<} \subseteq R_{<} \;\wedge\; S_{>} \subseteq R_{>}) \\
= & \{ \quad \text{rearranging and anti-symmetry of } \subseteq \quad \} \\
 & R_{<} = S_{<} \;\wedge\; R_{>} = S_{>} \quad .
\end{array}$$

So

$$\begin{array}{ll}
 & R \preceq S \;\wedge\; S \preceq R \\
\Rightarrow & \{ \quad \text{definition 53 and above} \quad \} \\
 & R = R_{<} \circ S \circ R_{>} \;\wedge\; R_{<} = S_{<} \;\wedge\; R_{>} = S_{>} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & R = S_{<} \circ S \circ S_{>} \\
= & \{ \quad \text{domains} \quad \} \\
 & R = S \quad .
\end{array}$$

□

The definition of "minimal" and "maximal" with respect to the thins relation on arbitrary relations is the same as definition 22 except that the dummies in the universal quantifications range over arbitrary relations (of appropriate type).

We recall the definition of a core relation [BV22, BV23b].

Definition 55 (Core Relation) A relation R is a *core relation* iff $R_{<} = R_{\prec}$ and $R_{>} = R_{\succ}$.

□

Lemma 56 A core relation is minimal with respect to the thins ordering on arbitrary relations.

Proof Suppose S is a core relation. Then, for all R,

$$
\begin{array}{ll}
 & R \preceq S \\
= & \{ \quad \text{definition 53} \quad \} \\
 & R_{\prec} \preceq S_{\prec} \;\wedge\; R_{\succ} \preceq S_{\succ} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
\Rightarrow & \{ \quad \text{assumption: } S \text{ is a core relation i.e. } S_{<} = S_{\prec} \text{ and } S_{>} = S_{\succ}\,, \\
 & \quad\;\; \text{so, by lemma 23, } S_{\prec} \text{ and } S_{\succ} \text{ are minimal; definition 22} \quad \} \\
 & R_{\prec} = S_{\prec} \;\wedge\; R_{\succ} = S_{\succ} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
= & \{ \quad S_{<} = S_{\prec} \text{ and } S_{>} = S_{\succ}\,, \text{ Leibniz} \quad \} \\
 & R_{\prec} = S_{<} \;\wedge\; R_{\succ} = S_{>} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
\Rightarrow & \{ \quad \text{Leibniz} \quad \} \\
 & (R_{\prec})_{<} = (S_{<})_{<} \;\wedge\; (R_{\succ})_{>} = (S_{>})_{>} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
= & \{ \quad \text{per domains and domains} \\
 & \quad\;\; \text{(specifically, for left domains: } [\,(R_{\prec})_{<} = R_{<}\,]\,,\ [\,(R_{<})_{<} = R_{<}\,]\,, \\
 & \quad\;\; \text{similarly for right domains)} \quad \} \\
 & R_{<} = S_{<} \;\wedge\; R_{>} = S_{>} \;\wedge\; R = R_{<} \circ S \circ R_{>} \\
\Rightarrow & \{ \quad \text{Leibiz and domains} \quad \} \\
 & R = S \ .
\end{array}
$$

Thus, by definition, S is minimal with respect to the thins ordering on arbitrary relations.
□

For reference, we include the definition of an index of an arbitrary relation and several of its properties. Proofs are given in [BV22, BV23b].

Definition 57 (Index) An *index* of a relation R is a relation J that has the following properties:

(a) $J \subseteq R$,

(b) $R_{\prec} \circ J \circ R_{\succ} = R$,

(c) $J_{<} \circ R_{\prec} \circ J_{<} = J_{<}$,

(d) $J_{>} \circ R_{\succ} \circ J_{>} = J_{>}$.

□

Lemma 58 If J is an index of the relation R then

$$J_{\prec} \subseteq R_{\prec} \;\wedge\; J_{\succ} \subseteq R_{\succ} \ .$$

It follows that

$$J_{<} = J_{\prec} \;\wedge\; J_{>} = J_{\succ} \ .$$

That is, an index is a core relation.
□

Lemma 59 Suppose J is an index of R. Then

(a) $R_{\prec} \circ J_{<} \circ R_{\prec} = R_{\prec}$,

(b) $R_{\succ} \circ J_{>} \circ R_{\succ} = R_{\succ}$.

□

Theorem 60 Suppose J is an index of R. Then $J_{<}$ is an index of $R_{\prec}$ and $J_{>}$ is an index of $R_{\succ}$.

□

We now resume the study of the extended thins ordering.

Lemma 61 If J is an index of R then $J \preceq R$.

Proof Suppose that J is an index of R. By definition 53, we have to prove that $J_{\prec} \preceq R_{\prec}$, $J_{\succ} \preceq R_{\succ}$ and $J = J_{<} \circ R \circ J_{>}$. For the first property, we have:

$J_{\prec} \preceq R_{\prec}$

$=$ { definition 5 }

$J_{\prec} = (J_{\prec})_{\asymp} \circ R_{\prec} \circ (J_{\prec})_{\asymp} \quad \wedge \quad R_{\prec} = R_{\prec} \circ (J_{\prec})_{\asymp} \circ R_{\prec}$

$=$ { domains (specifically $[\,(R_{\prec})_{\asymp} = R_{<}\,]$ with $R := J$) }

$J_{\prec} = J_{<} \circ R_{\prec} \circ J_{<} \quad \wedge \quad R_{\prec} = R_{\prec} \circ J_{<} \circ R_{\prec}$

$=$ { J is an index of R: theorem 60 and definition 57(c) with $J,R := J_{<}, R_{\prec}$;
J is an index of R: lemma 59(a) }

$J_{\prec} = J_{<} \quad \wedge \quad \text{true}$

$=$ { lemma 58 }

true .

By symmetry, $J_{\succ} \preceq R_{\succ}$. The third property is straightforward:

$J_{<} \circ R \circ J_{>}$

$=$ { assumption: J is an index of R, definition 57(b) }

$J_{<} \circ R_{\prec} \circ J \circ R_{\succ} \circ J_{>}$

$=$ { domains }

$J_{<} \circ R_{\prec} \circ J_{<} \circ J \circ J_{>} \circ R_{\succ} \circ J_{>}$

$=$ { assumption: J is an index of R, definition 57(c) and 57(d) }

$J_{<} \circ J \circ J_{>}$

$=$ { domains }

J .

□

Lemma 62 If relation S is minimal with respect to the thins ordering on arbitrary relations and J is an index of S then $J = S$.

Proof Immediate from lemma 61 and the definition of minimal.

□

Lemma 63 Assuming axiom 3 (our axiom of choice), if relation S is minimal with respect to the thins ordering on arbitrary relations then $S_{\prec}$ and $S_{\succ}$ are mimimal with respect to the thins ordering on pers.

Proof

S is minimal
$\Rightarrow$ { lemma 62 (assuming axiom 3) }
S is an index of S
$\Rightarrow$ { lemma 58 and theorem 60 }
$S_{<} = S_{\prec} \;\wedge\; S_{<}$ is an index of $S_{\prec}$
$\Rightarrow$ { lemma 23 }
$S_{\prec}$ is minimal .

Symmetrically, if S is minimal then $S_{\succ}$ is minimal.
□

Note that the axiom of choice is invoked in the proof of lemma 63: the application of lemma 62 in the first step assumes that S has an index and that this is so is a consequence of the axiom of choice [BV22, BV23b]. Consequently, the axiom of choice is also required in the statement and proof of the main theorem of this section:

Theorem 64 Assuming axiom 3 (our axiom of choice), a relation S is minimal with respect to the thins ordering on arbitrary relations iff S is a core relation.

Proof "If" is lemma 56. "Only if" is a combination of lemma 63 and theorem 24:

S is minimal
$\Rightarrow$ { lemma 63 (assuming axiom 3) }
$S_{\prec}$ is minimal
$\Rightarrow$ { theorem 24 }
$S_{\prec} \subseteq \mathbb{I}$
$=$ { domains }
$(S_{\prec})_{<} = S_{\prec}$
$=$ { per domains }
$S_{<} = S_{\prec}$.

That is,

$$S \text{ is minimal} \;\Rightarrow\; S_{<} = S_{\prec} \;.$$

Symmetrically,

$$S \text{ is minimal} \;\Rightarrow\; S_{>} = S_{\succ} \;.$$

Thus, by definition 55,

$$S \text{ is minimal} \;\Rightarrow\; S \text{ is a core relation} \;.$$

□

8 Conclusion

For us, the primary purpose of point-free relation algebra is to enable precise and concise reasoning about binary relations. Therefore, the axiomatisation does not capture all the properties of concrete relations and sometimes it is necessary to add axioms in order to facilitate such reasoning. Earlier work [BV22, BV23b] focused on facilitating pointwise reasoning —whilst not compromising the concision and precision of point-free reasoning— by adding axioms expressing the powerset properties of relations of a given type. To this end, [BV22, BV23b] proposed an axiom of choice (axiom 3) together with a saturation axiom (the axiom that the lattice of coreflexives of a given type is saturated by points). In this paper, the focus is on just the axiom of choice.

A central contribution is to provide further insight into the notion of a core relation introduced in [BV22, BV23b]. Theorem 64 shows that the relations that are minimal with respect to the thins relation are precisely the core relations.

The most challenging aspect of the paper has been the proof of theorem 52, which characterises pers that are maximal with respect to the thins ordering. In meeting this challenge, a major contribution is the introduction of a new idiom to point-free relation algebra that avoids the case analysis on whether or not a relation is empty. In formal terms, we avoid appeals to the cone rule. As a consequence, we extend the validity of the characterisation to models quite different from concrete relations. It remains to be seen whether or not this will be beneficial in practical applications.

The challenge we imposed on ourselves has undoubtedly increased the length of the proof of theorem 52 considerably: several of the lemmas (for example, lemma 44) are trivial if the cone rule is assumed. The fact that we have overcome the challenge attests to the strength of our axiom of choice.

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References

[BV22] Roland Backhouse and Ed Voermans. The index and core of a relation. With application to the axiomatics of relation algebra and block-ordered relations. Available at ResearchGate, April 2022.

[BV23a] Roland Backhouse and Ed Voermans. Diagonals and block-ordered relations. Submitted for publication. Available at ResearchGate and http://arXiv.org/abs/2401.17130, September 2023.

[BV23b] Roland Backhouse and Ed Voermans. The index and core of a relation. With application to the axiomatics of relation algebra. Submitted for publication. Available at ResearchGate and http://arXiv.org/abs/2309.02017, September 2023.

[Fv90] P.J. Freyd and A. Ščedrov. *Categories, Allegories*. North-Holland, 1990.

[Voe99] Ed (Theodorus Sebastiaan) Voermans. *Inductive Datatypes with Laws and Subtyping. A Relational Model.* Technische Universiteit Eindhoven, Department of Mathematics and Computer Science, 1999. Ph.D. thesis. DOI:10.6100/IR51.