The Schröder-Bernstein Theorem

Roland Backhouse

16th December 1993

In a paper I received from Martin Simons the following theorem is verified.

Let X and Y be sets and suppose there exist one-to-one maps $f \in X \to Y$ and $g \in Y \to X$. Then there exists a bijective map h from X onto Y.

The proof Martin sent me uses a lemma ("Banach Decomposition") which comes out of the blue. I wanted to see if I could *construct* a proof of the theorem — rather than *verify* the theorem. This note describes the outcome.

First, I found it beneficial to translate the theorem into the algebra of relations. Given are two relations f and g. Both are functional and injective, and the range of the one is contained in the domain of the other. That is,

(1) $f^{\cup} \circ f = \text{rng}.f \sqsubseteq \text{dom}.g = g \circ g^{\cup}$

and

(2) $g^{\cup} \circ g = \text{rng}.g \sqsubseteq \text{dom}.f = f \circ f^{\cup}$.

(The condition $f^{\cup} \circ f = \text{rng}.f$ expresses that f is functional – from left to right –, the condition $\text{dom}.g = g \circ g^{\cup}$ that g is injective.)

Required is to construct two relations h and k such that

(3) $h = k^{\cup}$

(4) $h \circ k = \text{dom}.f$

(5) $k \circ h = \text{dom}.g$.

This then is the problem. Apart from reformulating it in the algebra of relations I have also introduced the function k for the simple reason that by doing so the symmetry between f and g remains intact.

Using (2) and (3) we can rewrite (4) into the form

(6) $h \circ h^{\cup} = f \circ f^{\cup}$.

Similarly, (5) can be rewritten into the form

(7) $k \circ k^{\cup} = g \circ g^{\cup}$.

The obvious assignments $h{:=}f$ and $k{:=}g$ do not however necessarily satisfy (3). On the other hand, the assignments $h{:=}f$ and $k{:=}f^{\cup}$ satisfy (4) and (6) but not necessarily (7). Dually, the assignments $h{:=}g$ and $k{:=}g^{\cup}$ satisfy (4) and (7) but not necessarily (6). The solution would thus seem to be to assign to h some combination of f and $g^{\cup}$, and to k some combination of $f^{\cup}$ and g. Let us therefore introduce monotypes ("guards") A and B and postulate

(8) $h \;=\; A{\circ}f \sqcup \neg A \circ g^{\cup}$

and

(9) $k \;=\; B{\circ}g \sqcup \neg B \circ f^{\cup}$.

(Note that the form of these postulates has been carefully chosen so as to retain the symmetry in the problem.)

Now we try to solve (3) through (9). We begin with (3).

$$h = k^{\cup}$$

$\equiv$ { (8) and (9), converse }

$$A{\circ}f \sqcup \neg A \circ g^{\cup} \;=\; g^{\cup} \circ B \sqcup f \circ \neg B$$

$\Leftarrow$ { Leibniz and commutativity of $\sqcup$ }

$$A{\circ}f = f \circ \neg B \;\wedge\; \neg A \circ g^{\cup} = g^{\cup} \circ B$$

$\equiv$ { converse }

$$A{\circ}f = f \circ \neg B \;\wedge\; B{\circ}g = g \circ \neg A \;.$$

Thus we demand

(10) $A{\circ}f = f \circ \neg B$

and

(11) $B{\circ}g = g \circ \neg A$.

The follows-from step above may seem to be very coarse, but it is justified by the fact that the only known relationship between f and g is the relationship between their domains. In the next step we use some elementary domain calculus.

$$A{\circ}f = f \circ \neg B$$

$\equiv$ { domain translation (exploiting $dom.f = f \circ f^{\cup}$) }

$$f \circ rng.(A{\circ}f) = f \circ \neg B$$

$\Leftarrow$ { Leibniz }

$$rng.(A{\circ}f) = \neg B \quad .$$

Symmetrically,

$$B{\circ}g = g \circ \neg A \;\Leftarrow\; rng.(B{\circ}g) = \neg A \quad .$$

We therefore replace (10) and (11) by

(12) $rng.(A{\circ}f) = \neg B$

(13) $rng.(B{\circ}g) = \neg A$.

The key step is to observe that (12) and (13) do have (simultaneous) solutions in the unknowns A and B. Specifically, by eliminating B we obtain the requirement on A

$$rng.(\neg(rng.(A{\circ}f)) \circ g) = \neg A \quad .$$

But the function $A \mapsto \neg(rng.(\neg(rng.(A{\circ}f)) \circ g))$ is monotonic (since rng is monotonic and $\neg$ is anti-monotonic). Moreover, the relations (of a given type) form a complete lattice under the usual subset ordering. The function thus has a fixed point – courtesy of the Knaster-Tarski theorem. Equally, the function $B \mapsto \neg(rng.(\neg(rng.(B{\circ}g)) \circ f))$ also has a fixed point, and substituting the two fixed points for A and B, respectively, we obtain a solution to the two requirements (12) and (13).

It remains to see whether (12) and (13) automatically guarantee (4) and (5). By symmetry it is sufficient to check (4). Recalling that (12) and (13) are implied by (10) and (11), we have:

$$h{\circ}k$$

$=$ { (8) and (9) }

$$(A{\circ}f \sqcup \neg A \circ g^{\cup}){\circ}(\neg B \circ f^{\cup} \sqcup B{\circ}g)$$

$=$ { distributivity }

$$A \circ f \circ \neg B \circ f^{\cup} \;\sqcup\; A{\circ}f{\circ}B{\circ}g \;\sqcup\; \neg A \circ g^{\cup} \circ \neg B \circ f^{\cup} \;\sqcup\; \neg A \circ g^{\cup} \circ B \circ g$$

$=$ { (10) and (11) }

$$A \circ f \circ (A{\circ}f)^{\cup} \;\sqcup\; f \circ \neg B \circ B \circ g \;\sqcup\; g^{\cup} \circ B \circ \neg B \circ f^{\cup} \;\sqcup\; (B{\circ}g)^{\cup} \circ B \circ g$$

$=$ { domain calculus and (2); monotypes;
monotypes; range calculus and (7) }

$\mathsf{dom}.(A \circ \mathsf{f}) \sqcup \perp\!\!\perp \sqcup \perp\!\!\perp \sqcup \mathsf{rng}.(B \circ \mathsf{g})$

$=$ { calculus }

$\mathsf{dom}.(A \circ \mathsf{f}) \sqcup \mathsf{rng}.(B \circ \mathsf{g})$

$=$ { domain calculus; }

$A \circ \mathsf{dom.f} \sqcup \mathsf{rng}.(B \circ \mathsf{g})$

$=$ { $\mathsf{rng}.(B \circ \mathsf{g}) \sqsubseteq \{\text{monotonicity}\}\, \mathsf{rng.g} \sqsubseteq \{(2)\}\, \mathsf{dom.f}$ }

$A \circ \mathsf{dom.f} \sqcup \mathsf{rng}.(B \circ \mathsf{g}) \circ \mathsf{dom.f}$

$=$ { (13) and calculus }

$\mathsf{dom.f}$.

This completes the proof.