Generic Termination

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Outline

- Well-founded and Admits-Induction
- Hylomorphisms
- F-well-founded and F-inductive
- (Some) rules for reductivity
- Conclusions

Well-Founded, Admits Induction

Monotype (coreflexive, proposition)	$A,\ B,\ C$
Relation	$R,\ S,\ T$
empty, universal, identity relation	$\perp\!\!\!\perp,\ \top\!\!\!\top,\ \mathsf{id}$
Converse	$R^{\cup}$
Left and right domains (range and domain)	$R^{<},\ R^{>}$
Composition of relations	$R \cdot S$
Weakest subspecification	$R \backslash S$
Weakest (liberal) precondition	$R \backslash_{\bullet} A$
Composition of functions (on relations)	$f \circ g$
admits induction	$\mu(R \backslash_{\bullet}) = \mathsf{id}$
admits induction	$\mu(R\backslash) = \top\!\!\!\top$
well-founded	$\nu(\,\cdot R) = \perp\!\!\!\perp$
well-founded	$\nu\langle A \mapsto (A \cdot R)^{>} \rangle = \perp\!\!\!\perp$

$$\neg(\nu(\,\cdot R)) \;=\; (\mu(R\backslash))^{\cup} \quad .$$

Relator

A *relator* F is a function to the objects of an allegory $\mathcal{C}$ from the objects of an allegory $\mathcal{D}$ together with a mapping to the arrows (relations) of $\mathcal{C}$ from the arrows of $\mathcal{D}$ satisfying the following properties:

$$\mathsf{F.R} :: \mathsf{F.I} \xleftarrow{\mathcal{C}} \mathsf{F.J} \qquad \text{whenever } \mathsf{R} :: \mathsf{I} \xleftarrow{\mathcal{D}} \mathsf{J} \ .$$

$$\mathsf{F.R} \cdot \mathsf{F.S} = \mathsf{F.(R \cdot S)} \qquad \text{for each } \mathsf{R} \text{ and } \mathsf{S} \text{ of composable type,}$$

$$\mathsf{F.id}_A = \mathsf{id}_{\mathsf{F}.A} \qquad \text{for each object } A \ ,$$

$$\mathsf{F.R} \subseteq \mathsf{F.S} \Leftarrow \mathsf{R} \subseteq \mathsf{S} \qquad \text{for each } \mathsf{R} \text{ and } \mathsf{S} \text{ of the same type,}$$

$$(\mathsf{F.R})^{\cup} = \mathsf{F.(R^{\cup})} \qquad \text{for each } \mathsf{R} \ .$$

The Hylo Theorem

Definition 1 Assume that F is an endorelator. Then $(\mathrm{I}\,,\mathsf{in})$ is a *relational initial* F-algebra iff $\mathsf{in} :: \mathrm{I} \leftarrow \mathsf{F}.\mathrm{I}$ and there is a mapping $(\!|\_|\!)$ defined on all F-algebras such that

$$(\!|R|\!) :: A \leftarrow \mathrm{I} \qquad \text{if } R :: A \leftarrow \mathsf{F}.A$$

$$(\!|\mathsf{in}|\!) = \mathsf{id}_{\mathrm{I}}$$

$$(\!|R|\!) \cdot (\!|S|\!)^{\cup} = \mu\langle X \mapsto R \cdot \mathsf{F}.X \cdot S^{\cup}\rangle$$

□

Theorem 2 (Hylo Theorem) Suppose F is an endorelator on a locally-complete, tabular allegory $\mathcal{A}$. Let F' denote the endofunctor obtained by restricting F to the objects and arrows of $\mathsf{Map}(\mathcal{A})$. Then in is an initial F'-algebra if and only it is a relational initial F-algebra.

□

Hylo Programs

$$\begin{aligned}
\mathsf{fact} &= \mathsf{one} \triangledown (\mathsf{times} \cdot \mathsf{succ} \times \mathsf{id}_{\mathsf{Nat}}) \quad\cdot\quad \mathsf{id}_{\mathbb{1}} + (\mathsf{id}_{\mathsf{Nat}} \vartriangle \mathsf{fact}) \quad\cdot\quad \mathsf{zero}^{\cup} \blacktriangledown \mathsf{succ}^{\cup} \\
\mathsf{suffix} &= \mathsf{nil} \triangledown ((\mathsf{cons} \cdot \mathsf{exl}) \cup (\mathsf{exr} \cdot \mathsf{exr})) \quad\cdot\quad \mathsf{id}_{\mathbb{1}} + (\mathsf{id}_{I} \times (\mathsf{id}_{\mathsf{List}.I} \vartriangle \mathsf{suffix})) \quad\cdot\quad \mathsf{nil}^{\cup} \blacktriangledown \mathsf{cons}^{\cup} \\
\mathsf{qs} &= \mathsf{nil} \triangledown (\mathsf{join} \cdot \mathsf{id}_{I} \times \mathsf{cons}) \quad\cdot\quad \mathsf{id}_{\mathbb{1}} + (\mathsf{qs} \times (\mathsf{id}_{I} \times \mathsf{qs})) \quad\cdot\quad \mathsf{nil}^{\cup} \blacktriangledown \mathsf{dnf} \\
X &= R \triangledown \mathsf{conquer} \quad\cdot\quad \mathsf{id}_{I} + (X \times X) \quad\cdot\quad \mathsf{id}_{I} + \mathsf{divide} \quad\cdot\quad A \blacktriangledown B \\
\mathsf{do} &= \mathsf{id}_{I} \triangledown \mathsf{id}_{I} \quad\cdot\quad \mathsf{id}_{I} + \mathsf{do} \quad\cdot\quad \sim B \blacktriangledown (S \cdot B) \\
L &= (\mathsf{concat} \cdot a \times \mathsf{id} \times b) \triangledown c \quad\cdot\quad (a \times L \times b) + c \quad\cdot\quad (a \times \mathsf{id} \times b \cdot \mathsf{concat}^{\cup}) \blacktriangledown c \\
\mathsf{slsrt} &= \mathsf{nil} \triangledown \mathsf{cons} \quad\cdot\quad \mathsf{id}_{\mathbb{1}} + \mathsf{id}_{I} \times \mathsf{slsrt} \quad\cdot\quad \mathsf{nil}^{\cup} \blacktriangledown (\mathsf{cons}^{\cup} \cdot \mathsf{select}) \\
\mathsf{join} &= \mathsf{post} \quad\cdot\quad (\mathsf{id}_{\mathbb{1}} + (\mathsf{id}_{I} \times \mathsf{join})) \times \mathsf{id}_{\mathsf{List}.I} \quad\cdot\quad \mathsf{pass} \vartriangle \mathsf{exr} \quad\cdot\quad (\mathsf{nil}^{\cup} \blacktriangledown \mathsf{cons}^{\cup}) \times \mathsf{id}_{\mathsf{List}.I} \\
\mathsf{fib} &= \mathsf{zero} \triangledown \mathsf{one} \triangledown \mathsf{add} \cdot \mathsf{id} + \mathsf{id} + (\mathsf{fib} \times \mathsf{fib}) \cdot \mathsf{id} + \mathsf{id} + (\mathsf{id} \vartriangle \mathsf{succ}) \cdot \mathsf{zero}^{\cup} \blacktriangledown \mathsf{one}^{\cup} \blacktriangledown (\mathsf{succ}^{2})^{\cup}
\end{aligned}$$

Generalisations of wf and admits-induction

Relation $\mathsf{R} :: \mathsf{F.I} \leftarrow \mathsf{I}$ is F-*well-founded* iff, for all relations $\mathsf{S} :: \mathsf{I} \leftarrow \mathsf{F.I}$,

$$\nu\langle \mathsf{X} \mapsto \mathsf{R} \cdot \mathsf{F.X} \cdot \mathsf{S}^{\cup} \rangle \;\; = \;\; \mu\langle \mathsf{X} \mapsto \mathsf{R} \cdot \mathsf{F.X} \cdot \mathsf{S}^{\cup} \rangle \quad .$$

A relation $\mathsf{R} :: \mathsf{I} \leftarrow \mathsf{F.I}$ is F-*inductive* iff

$$\nu\langle A \mapsto (\mathsf{R} \cdot \mathsf{F}.A)^{<} \rangle = \mathsf{id}_{\mathrm{I}} \quad .$$

Relation $\mathsf{R} :: \mathsf{F.I} \leftarrow \mathsf{I}$ is F-*reductive* iff

$$\mu\langle A \mapsto \mathsf{R} \backslash \mathsf{F}.A \rangle \; = \; \mathsf{id}_{\mathrm{I}}$$

Reducing Problem Size

Relation $\mathsf{mem} :: \mathrm{I} \leftarrow \mathsf{F}.\mathrm{I}$ is a *membership* relation of relator F if and only if it satisfies, for all coreflexives A, $A \subseteq \mathrm{I}$:

$$\mathsf{F}.A = \mathsf{mem} \setminus\!\!\!\!\cdot\, A \quad .$$

Pointwise:

$$xs \in \mathsf{F}.A \;\equiv\; \forall(x{:}\; x\langle\mathsf{mem}\rangle xs{:}\; x{\in}A) \quad .$$

Theorem (Hoogendijk and De Moor):

$$R \text{ is } \mathsf{F}\text{-reductive} \;\equiv\; \mathsf{mem}\cdot R \text{ is well-founded} \quad .$$

Basic F-reductive relations

Theorem The converse of an initial F-algebra is F-reductive.

Corollary The cata program

$$X = R \cdot F.X \cdot in^{\cup}$$

is terminating.

Theorem Let $\oplus$ be a binary relator, in_{I} an initial $(I\oplus)$-algebra, and T the tree relator corresponding to $\oplus$ and in_{I}. Then $in_{I}{}^{\cup} \cdot T.\top\!\!\top_{I\leftarrow I}$ is $(I\oplus)$-reductive.

Corollary Selection sort

$$slsrt = nil \triangledown cons \cdot \mathbb{1}{+}I{\times}slsrt \cdot nil^{\cup} \blacktriangledown (cons^{\cup} \cdot select)$$

is terminating.

Proof

$$nil^{\cup} \blacktriangledown (cons^{\cup} \cdot select) \subseteq nil^{\cup} \blacktriangledown cons^{\cup} \cdot List.\top\!\!\top \ .$$

New From Old

Theorem Suppose $R :: F.I \leftarrow I$ is F-reductive. Define the function f on positive numbers by $f.1 = R$, $f.(n+1) = F.(f.n) \cdot R$. Then $f.n$ is F^n-reductive.

Example The fibonacci program

$$\mathsf{fib} \;\; = \;\; \mathsf{zero} \triangledown \mathsf{one} \triangledown \mathsf{add} \cdot \mathsf{id}{+}\mathsf{id}{+}(\mathsf{fib} \times \mathsf{fib}) \cdot \mathsf{id}{+}\mathsf{id}{+}(\mathsf{id} \vartriangle \mathsf{succ}) \cdot \mathsf{zero}^{\cup} \blacktriangledown \mathsf{one}^{\cup} \blacktriangledown (\mathsf{succ}^2)^{\cup}$$

is terminating.

New From Old

Theorem Suppose $R :: F.I \leftarrow I$ is F-reductive, $S :: H.(G.I) \leftarrow G.(F.I)$ is such that $S :: H{\circ}G \stackrel{.}{\hookleftarrow} G{\circ}F$, and G is a relator that is a lower adjoint in a Galois connection. Then $S \cdot G.R$ is H-reductive.

Examples

$$0{+}n = n \quad \text{and} \quad (m{+}1){+}n = (m{+}n){+}1$$
$$0{\times}n = 0 \quad \text{and} \quad (m{+}1){\times}n = m{\times}n + n$$
$$n^0 = 1 \quad \text{and} \quad n^{m+1} = n^m{\times}n$$
$$\mathsf{nil} +\!\!+\, ys = ys \quad \text{and} \quad (x\ xs) +\!\!+\, ys = x\ (xs +\!\!+\, ys)$$

Generically:

$$X = R \cdot F.X \times P \cdot F.(I{\times}S) \times P \cdot \mathsf{pass} \triangle \mathsf{exr} \cdot \mathsf{in}^{\cup} \times P$$

where $\mathsf{pass} \cdot F.A \times P \subseteq F.(A{\times}P) \cdot \mathsf{pass}$.

I.e. $\mathsf{pass} :: F \circ (\times P) \stackrel{.}{\hookleftarrow} (\times P) \circ F$

Hence $F.(P{\times}S) \times P \cdot \mathsf{pass} \triangle \mathsf{exr} :: (\times P) \circ F \circ (\times P) \stackrel{.}{\hookleftarrow} (\times P) \circ F$

and $F.(I{\times}S) \times P \cdot \mathsf{pass} \triangle \mathsf{exr} \cdot \mathsf{in}^{\cup} \times P$ is $(\times P) \circ F$ reductive.

New From Old

Corollary If R is F-reductive and $\mathsf{S} :: \mathsf{H} \stackrel{\cdot}{\leftarrow} \mathsf{F}$ then $\mathsf{S} \cdot \mathsf{R}$ is H-reductive.

Theorem Let Q be G-reductive and $\mathsf{S} :: \mathsf{F} \stackrel{\cdot}{\leftarrow} \mathsf{Id}$, where Id denotes the identity relator. Then $\mathsf{F.Q} \cdot \mathsf{S}$ is $(\mathsf{F} \circ \mathsf{G})$-reductive.

Proof Follows from:

$$\mu(A \mapsto \mathsf{Q} \setminus \mathsf{G}.A) \;\subseteq\; \mu(A \mapsto (\mathsf{F.Q} \cdot \mathsf{S}) \setminus \mathsf{F}.(\mathsf{G}.A)) \quad .$$

Conclusions

- Discipline of (recursive) programming based on virtual data structures.
- Introduction of explicit parameter encourages analysis of dependance on the structure of the parameter.
- Proof of termination akin to type checking.

Reference: Henk Doornbos, ”Reductivity arguments and program construction”, PhD thesis (1996). Available at `http://www.cs.nott.ac.uk/~rcb/papers`.