When is a function a fold or an unfold?

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1. A problem

Consider the following two functions:

```
> either x y zs = elem x zs || elem y zs
> both   x y zs = elem x zs && elem y zs
```

where

```
> elem x = foldr ((||).(x==)) False

> foldr f e []     = e
> foldr f e (x:xs) = f x (foldr f e xs)
```

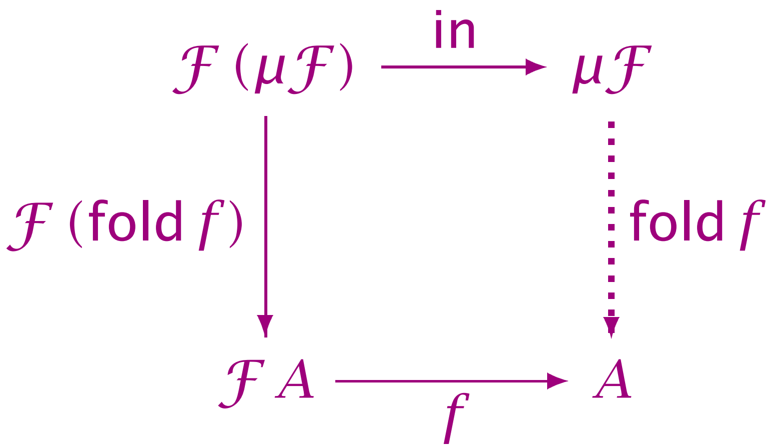
Can `either` or `both` be expressed directly as a `foldr`?
(They would be more efficient that way.)

2. Folds

Categorically speaking, an *algebra* for a functor $\mathcal{F}$ is a pair (A, f) with

$$\mathcal{F}\,A \xrightarrow{\;f\;} A$$

Initial algebra $(\mu\mathcal{F}, \mathsf{in})$ for functor $\mathcal{F}$ has unique homomorphism to any other such algebra:



For instance, with $\mathcal{F}\,X = 1 + \mathit{Nat} \times X$, initial algebra $\mu\mathcal{F}$ is finite lists of naturals. Function *sum* is an example of a fold.

3. The question

Which h can be written as a fold?

That is, which h can be written in the form

$$h = \mathsf{fold}\, f$$

for some f of the appropriate type?

4. Non-answers

Universal property states that

$$h = \mathsf{fold}\, f \quad \Leftrightarrow \quad h \circ \mathsf{in} = f \circ \mathcal{F}\, h$$

This is not such a satisfactory answer, as it entails knowing f, an *intensional* aspect of h.

Moreover, such an f is not always obvious, even when one does exist.

4.1. Another non-answer: Injectivity

Partial answer, but purely *extensional*: h in *Set* can be written as a fold if it is injective.

For if h is injective, then there exists g with $g \circ h = \mathsf{id}$, and

$$h = \mathsf{fold}\,(h \circ \mathsf{in} \circ \mathcal{F}\,g)$$

For example, *rev* is injective, so is a fold.

(Corollary: for any f, the h such that $h\,x = (x, f\,x)$ is a fold.)

An extensional answer, because depends only on observable aspects of h.

Only a partial answer, because only an implication. For example, *sum* is not injective, yet is a fold.

4.2. More non-answers: Fusion etc

More extensional but still partial answers: h of the form

- $\mathsf{fold}\, f \circ \mathsf{map}\, g$
- $g \circ \mathsf{fold}\, f$ (provided $g \circ f = f' \circ \mathcal{F}\, g$ for some f')
- $\mathit{fork}\,(\mathsf{fold}\, f, \mathsf{fold}\, g)$

can be written as a fold.

Still no complete answer, even when all taken together.
We want an *equivalence*.

5. Main theorem for folds

Characterization as fold boils down to properties of *congruences* and *kernels*.

Ugly proofs in $\mathcal{S}et$ and $\mathcal{P}fun$.

Elegant proof for total functions in $\mathcal{R}el$.

5.1. Congruences

Given relation $S : \mathcal{F}\,A \leadsto A$, say that relation $R : A \leadsto A$ is an *$\mathcal{F}$-congruence for S* when

$$S \circ \mathcal{F}\,R \subseteq R \circ S$$

Informally, arguments to S related (pointwise under $\mathcal{F}$) by R will yield results from S related (directly) by R.

(When R is an ordering, R is an $\mathcal{F}$-congruence for S iff S is monotonic under R. But we will be using this for non-ordering Rs.)

5.2. Kernels

Define the *kernel* of a relation R by

$$\mathsf{ker}\, R = R^{\circ} \circ R$$

5.3. Theorem for folds

Function $h : \mu\mathcal{F} \rightsquigarrow A$ (ie simple and entire relation) is a fold iff $\mathsf{ker}\, h$ is an $\mathcal{F}$-congruence for in.

5.4. Proof of theorem for folds

$$
\begin{array}{ll}
& \exists f.\quad h = \mathsf{fold}\, f \\
\Leftrightarrow & \quad \{\text{ folds }\} \\
& \exists f.\quad h \circ \mathsf{in} = f \circ \mathcal{F}\, h \\
\Leftrightarrow & \quad \{\text{ function equality as inclusion }\} \\
& \exists f.\quad h \circ \mathsf{in} \subseteq f \circ \mathcal{F}\, h \\
\Leftrightarrow & \quad \{\text{ shunting: } R \circ f^{\circ} \subseteq S \Leftrightarrow R \subseteq S \circ f \text{ }\} \\
& \exists f.\quad h \circ \mathsf{in} \circ \mathcal{F}\, h^{\circ} \subseteq f \\
\Leftrightarrow & \\
& h \circ \mathsf{in} \circ \mathcal{F}\, h^{\circ} \text{ is simple}
\end{array}
$$

Now. . .

$$
\begin{array}{ll}
 & h \circ \mathsf{in} \circ \mathcal{F}\, h^{\circ} \text{ is simple} \\
\Leftrightarrow & \quad \{ \text{simplicity} \} \\
 & (h \circ \mathsf{in} \circ \mathcal{F}\, h^{\circ}) \circ (h \circ \mathsf{in} \circ \mathcal{F}\, h^{\circ})^{\circ} \subseteq \mathsf{id} \\
\Leftrightarrow & \quad \{ \text{converse of composition} \} \\
 & h \circ \mathsf{in} \circ \mathcal{F}\, h^{\circ} \circ \mathcal{F}\, h \circ \mathsf{in}^{\circ} \circ h^{\circ} \subseteq \mathsf{id} \\
\Leftrightarrow & \quad \{ \text{shunting again, and dual: } f \circ R \subseteq S \Leftrightarrow R \subseteq f^{\circ} \circ S \} \\
 & \mathsf{in} \circ \mathcal{F}\, h^{\circ} \circ \mathcal{F}\, h \subseteq h^{\circ} \circ h \circ \mathsf{in} \\
\Leftrightarrow & \quad \{ \text{functors; kernels} \} \\
 & \mathsf{in} \circ \mathcal{F}\, (\mathsf{ker}\, h) \subseteq \mathsf{ker}\, h \circ \mathsf{in} \\
\Leftrightarrow & \quad \{ \text{congruences} \} \\
 & \mathsf{ker}\, h \text{ is an } \mathcal{F}\text{-congruence for } \mathsf{in}
\end{array}
$$

6. Examples of theorem

On finite lists of naturals, theorem reduces to: h is a fold iff kernel of h closed under *cons*:

$$h\,xs = h\,ys \quad \Rightarrow \quad h\,(cons\,(x, xs)) = h\,(cons\,(x, ys))$$

Kernel of *sum* is closed under *cons*, so *sum* is a fold.

Kernel of *stail* is not closed, where

$$\begin{array}{lcl} stail\,nil & = & nil \\ stail\,(cons\,(x, xs)) & = & xs \end{array}$$

so *stail* is not a fold.

6.1. Examples of theorem on trees

On finite binary trees

$$\mathit{Tree}\,A \quad = \quad \mathit{leaf}\ A + \mathit{node}\,(\mathit{Tree}\,A)\,(\mathit{Tree}\,A)$$

function h is a fold iff kernel of h closed under *node*:

$$h\,t = h\,t' \wedge h\,u = h\,u' \quad \Rightarrow \quad h\,(\mathit{node}\,(t,u)) = h\,(\mathit{node}\,(t',u'))$$

Kernel of $\mathit{bal} : \mathit{Tree}\,A \to \mathit{Bool}$ is not closed under *node*: even when (t,u) is in kernel, $(\mathit{node}\,(t,t), \mathit{node}\,(t,u))$ need not be. So *bal* is not a fold.

However, kernel of *dbal* such that $\mathit{dbal}\,t = (\mathit{depth}\,t, \mathit{bal}\,t)$ is closed under *node*, so *dbal* is a fold.

7. Duality

A *coalgebra* for a functor $\mathcal{F}$ is a pair (A, f) with

$$A \xrightarrow{\;f\;} \mathcal{F}\,A$$

Final coalgebra $(\nu\mathcal{F}, \mathsf{out})$ for functor $\mathcal{F}$ has unique homomorphism to any other such coalgebra:

$$\begin{array}{ccc}
A & \xrightarrow{\;f\;} & \mathcal{F}\,A \\
{\scriptstyle \mathsf{unfold}\,f}\,\big\downarrow & & \big\downarrow\,{\scriptstyle \mathcal{F}\,(\mathsf{unfold}\,f)} \\
\nu\mathcal{F} & \xrightarrow[\mathsf{out}]{} & \mathcal{F}\,(\nu\mathcal{F})
\end{array}$$

For instance, with $\mathcal{F}\,X = \mathit{Nat} \times X$, final coalgebra $\nu\mathcal{F}$ is streams of naturals. Function *from* such that $\mathit{from}\;n = [n, n+1, n+2, \ldots]$ is an example of an unfold.

7.1. Invariants

Given relation $S : A \rightsquigarrow \mathcal{F}\,A$, say that relation $R : A \rightsquigarrow A$ is an *$\mathcal{F}$-invariant for S* when

$$S \circ R \subseteq \mathcal{F}\,R \circ S$$

(Invariance is the dual of congruence.)

In particular, when R is a monotype ($R \subseteq \mathsf{id}$), applying S to arguments 'in' R yields results 'in' R (pointwise under $\mathcal{F}$).

7.2. Images

Define the *image* of a relation R by

$$\mathsf{img}\, R = R \circ R^{\circ}$$

(The image is the dual of the kernel.)

7.3. Theorem for unfolds

Function $h : A \rightsquigarrow \nu\mathcal{F}$ (ie simple entire relation) is an unfold iff $\mathsf{img}\, h$ is an $\mathcal{F}$-invariant for out.

Note that $\mathsf{img}\, h$ is a monotype.

7.4. Proof of theorem for unfolds

$$\begin{array}{ll}
& \exists f.\quad h = \mathsf{unfold}\, f \\
\Leftrightarrow & \quad \{\text{ unfolds }\} \\
& \exists f.\quad \mathsf{out} \circ h = \mathcal{F}\, h \circ f \\
\Leftrightarrow & \quad \{\text{ function equality as inclusion }\} \\
& \exists f.\quad \mathcal{F}\, h \circ f \subseteq \mathsf{out} \circ h \\
\Leftrightarrow & \quad \{\text{ shunting }\} \\
& \exists f.\quad f \subseteq \mathcal{F}\, h^{\circ} \circ \mathsf{out} \circ h \\
\Leftrightarrow & \\
& \mathcal{F}\, h^{\circ} \circ \mathsf{out} \circ h \text{ is entire}
\end{array}$$

Now...

$$
\begin{array}{ll}
 & \mathcal{F}\,h^{\circ} \circ \mathsf{out} \circ h \text{ is entire} \\
\Leftrightarrow & \quad \{ \text{entirety} \} \\
 & \mathsf{id} \subseteq (\mathcal{F}\,h^{\circ} \circ \mathsf{out} \circ h)^{\circ} \circ (\mathcal{F}\,h^{\circ} \circ \mathsf{out} \circ h) \\
\Leftrightarrow & \quad \{ \text{converse of composition} \} \\
 & \mathsf{id} \subseteq h^{\circ} \circ \mathsf{out}^{\circ} \circ \mathcal{F}\,h \circ \mathcal{F}\,h^{\circ} \circ \mathsf{out} \circ h \\
\Leftrightarrow & \quad \{ \text{shunting again} \} \\
 & \mathsf{out} \circ h \circ h^{\circ} \subseteq \mathcal{F}\,h \circ \mathcal{F}\,h^{\circ} \circ \mathsf{out} \\
\Leftrightarrow & \quad \{ \text{functors; images} \} \\
 & \mathsf{out} \circ \mathsf{img}\,h \subseteq \mathcal{F}\,(\mathsf{img}\,h) \circ \mathsf{out} \\
\Leftrightarrow & \quad \{ \text{invariants} \} \\
 & \mathsf{img}\,h \text{ is an } \mathcal{F}\text{-invariant for } \mathsf{out}
\end{array}
$$

7.5. Examples on lists

On streams of naturals, theorem reduces to: h is an unfold iff tail of a list produced by h may itself be produced by h:

$$\mathsf{img}\,(\mathit{tail} \circ h) \subseteq \mathsf{img}\,h$$

Now $\mathit{tail}\,(\mathit{from}\,n) = \mathit{from}\,(n+1)$, so from is an unfold.

But in general for no m is $\mathit{tail}\,(\mathit{mults}\,n) = \mathit{mults}\,m$, where

$$\mathit{mults}\,n = [0, n, 2 \times n, 3 \times n, \ldots]$$

so mults is not an unfold.

8. Back to original problem

Recall:

```
> either x y zs = elem x zs || elem y zs
> both   x y zs = elem x zs && elem y zs
```

Kernel of `either x y` is closed under cons, so `either` is a `foldr`:

```
either x y (z:zs) = (x==z) || (y==z) || either x y zs
```

Kernel of `both x y` is not closed under cons, so `both` is not a `foldr`:

```
both 1 2 [2] = False = both 1 2 [3]
both 1 2 (1:[2]) = True /= False = both 1 2 (1:[3])
```

9. Partiality

The results also hold (with suitable adaptations) for partial functions.

But I don't see (yet!) how to adapt the elegant relational proofs.

9.1. Set-theoretic version of main theorem for folds

Definition 1. Kernel $\mathsf{ker}\, f$ of $f : A \to B$ is the set of pairs identified by f:

$$\mathsf{ker}\, f = \{\, (a, a') \in A \times A \mid f\, a = f\, a' \,\}$$

Informally, it is necessary and sufficient for kernel of function to be 'closed under the constructors':

Theorem 2. Function $h : \mu\mathcal{F} \to A$ in Set is a fold iff

$$\mathsf{ker}\,(\mathcal{F}\, h) \subseteq \mathsf{ker}\,(h \circ \mathsf{in})$$

9.2. Lemmas for proof of Theorem 2

Crucial lemma — inclusion of kernels equivales existence of ‘postfactors’:

Lemma 3. For functions $f : A \to B$ and $h : A \to C$ in *Set*,

$$\exists g : B \to C.\quad h = g \circ f \quad \Leftrightarrow \quad \mathsf{ker}\, f \subseteq \mathsf{ker}\, h \wedge B \to C \neq \emptyset$$

Simple result about non-emptiness of algebra types:

Lemma 4.

$$\mu\mathcal{F} \to A \neq \emptyset \quad \Rightarrow \quad \mathcal{F}\, A \to A \neq \emptyset$$

9.3. Proof of Theorem 2

Almost embarrassingly simple:

$$\begin{aligned}
& \exists g.\quad h = \mathsf{fold}\, g \\
\Leftrightarrow \quad & \quad \{\text{universal property}\} \\
& \exists g.\quad h \circ \mathsf{in} = g \circ \mathcal{F}\, h \\
\Leftrightarrow \quad & \quad \{\text{Lemma 3}\} \\
& \ker(\mathcal{F}\, h) \subseteq \ker(h \circ \mathsf{in}) \quad \wedge \quad \mathcal{F}\, A \to A \neq \varnothing
\end{aligned}$$

Note that $h : \mu\mathcal{F} \to A$, so second conjunct follows from Lemma 4.

9.4. Generalizing for partial functions

Definition 5. Kernel $\mathsf{ker}\, f$ of partial function $f : A \to B$ is the equivalence relation

$$\begin{aligned}\mathsf{ker}\, f = {} & \{\, (a, a') \in A \times A \mid a, a' \in \mathsf{dom}\, f \wedge f\, a = f\, a' \,\} \cup \\ & \{\, (a, a') \in A \times A \mid a, a' \notin \mathsf{dom}\, f \,\}\end{aligned}$$

Lemma 6. For partial functions $f : A \to B$ and $h : A \to C$ in $\mathcal{P}fun$,

$$\exists g : B \to C.\quad h = g \circ f \quad \Leftrightarrow \quad \mathsf{ker}\, f \subseteq \mathsf{ker}\, h \wedge \mathsf{dom}\, f \supseteq \mathsf{dom}\, h$$

Theorem 7. Partial function $h : \mu\mathcal{F} \to A$ in $\mathcal{P}fun$ is a fold iff

$$\mathsf{ker}\,(\mathcal{F}\, h) \subseteq \mathsf{ker}\,(h \circ \mathsf{in}) \wedge \mathsf{dom}\,(\mathcal{F}\, h) \supseteq \mathsf{dom}\,(h \circ \mathsf{in})$$

9.5. Dualizing the generalization

Definition 8. Image $\mathsf{img}\,f$ of partial function $f : A \to B$ is the set

$$\mathsf{img}\,f = \{\, b \in B \mid \exists a \in \mathsf{dom}\,f. \quad f\,a = b \,\}$$

Lemma 9. For partial functions $f : B \to C$ and $h : A \to C$ in $\mathcal{P}fun$,

$$\exists g : A \to B. \quad h = f \circ g \quad \Leftrightarrow \quad \mathsf{img}\,f \supseteq \mathsf{img}\,h$$

Theorem 10. Partial function $h : A \to \nu\mathcal{F}$ in $\mathcal{P}fun$ is an unfold iff

$$\mathsf{img}\,(\mathcal{F}\,h) \supseteq \mathsf{img}\,(\mathsf{out} \circ h)$$