Slide 1

A Taste of Intuitionistic Type Theory
Part 2

Midland Graduate School, 18.10.00 - 15.11.00

`http://www.cs.nott.ac.uk/~txa/itt/`

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October 25, 2000

Slide 2

Reduction

$([x:A]a)b$	$\rhd$	$a[x:=b]$
$\text{fst (pair } a\ b)$	$\rhd$	a
$\text{snd (pair } a\ b)$	$\rhd$	b
$\text{case } ([x:A]a)\ ([y:B]b)\ (\text{inl } c)$	$\rhd$	$a[x:=c]$
$\text{case } ([x:A]a)\ ([y:B]b)\ (\text{inr } c)$	$\rhd$	$a[y:=c]$
$\text{existsE } ([x:A][y:P]q)\ (\text{existsI } a\ p)$	$\rhd$	$q[x:=a,y:=p]$

Slide 3

Proving weak normalisation

$$\text{WNF}(A) = \{t : A \mid \text{t weakly normal}\}$$

To show: $\forall t : a.\exists u \in \text{WNF}(A)$.

Define an interpretation of propositions:

$$\begin{aligned}
[\![A \to B]\!] &= \text{WNF}(A \to B) \times ([\![A]\!] \to [\![B]\!]) \\
[\![A \wedge B]\!] &= \text{WNF}(A \wedge B) \times ([\![A]\!] \times [\![B]\!]) \\
[\![A \vee B]\!] &= \text{WNF}(A \wedge B) \times ([\![A]\!] + [\![B]\!])
\end{aligned}$$

Define a sound interpretation of proofs: $\dfrac{p : A}{[\![p]\!] \in [\![A]\!]}$

And hence: $\dfrac{t : A}{\pi_1(A([\![p]\!]) \in \text{WNF}(A)}$

Slide 4

Function types

Formation

$$\frac{A : \mathbf{Set} \qquad B : \mathbf{Set}}{A \to B : \mathbf{Set}} \to \mathcal{F}$$

Introduction

$$\frac{\begin{array}{c}[x : A] \\ \vdots \\ b : B\end{array}}{[x : A]b : A \to B} \to \mathcal{I}$$

Elimination

$$\frac{f : A \to B \qquad a : A}{fa : B} \to \mathcal{E}$$

Slide 5

Computation

$$[([x:A]a)b = a[x:=b]$$

Slide 6

Propositional Equality

Formation

$$\frac{A : \mathbf{Set} \qquad a, b : A}{\mathrm{Id}\ a\ b : \mathbf{Prop}} = \mathcal{F}$$

Introduction

$$\frac{A : \mathbf{Set} \qquad a : A}{\mathrm{refl}\ a : \mathrm{Id}\ a\ a} = \mathcal{I}$$

Elimination

$$\frac{A : \mathbf{Set}\ a, b : A \qquad \begin{array}{c}[x:A]\\ \vdots\\ P\end{array} \qquad p : \mathrm{Id}\ a\ b \qquad q : P[x:=a]}{\mathrm{subst}\ ([x:A]Px)\ p\ q : P[x:=b]} = \mathcal{E}$$

Slide 7

Derivable properties of Id

$$\frac{A : \mathbf{Set} \qquad a, b : A \qquad p : \mathrm{Id}\ a\ b}{\mathrm{subst}\ ([x : A]\mathrm{Id}\ x\ a)\ p\ (\mathrm{refl}\ a) : \mathrm{Id}\ b\ a}$$

$$\frac{A : \mathbf{Set} \qquad a, b, c : A \qquad p : \mathrm{Id}\ a\ b \qquad q : \mathrm{Id}\ b\ c}{\mathrm{subst}\ ([x : A]\mathrm{Id}\ a\ x)\ q\ p : \mathrm{Id}\ a\ c}$$

$$\frac{A, B : \mathbf{Set} \qquad f : A \to B \qquad a, b : A \qquad p : \mathrm{Id}\ a\ b}{\mathrm{subst}\ ([x : A]\mathrm{Id}\ (fa)\ (fx))\ p\ (\mathrm{refl}\ (fa)) : \mathrm{Id}\ (fa)\ (fb)}$$

Slide 8

Definitions in LEGO

```
[symId = [A|Set][a,b|A][p:Id a b]
         subst ([x:A]Id x a) p (refl a)];

[transId = [A|Set][a,b,c|A][p:Id a b][q:Id b c]
           subst ([x:A]Id a x) q p];

[respId = [A,B|Set][f:A->B][a,b|A][p:Id a b]
          subst ([x:A]Id (f a) (f x)) p (refl (f a))];
```

Slide 9

Two equalities

defintional equality $t = u$
Decidable, built into LEGO.

propositional equality Id t u
Requires proof, has to be used explicitely.

Slide 10

Natural numbers

Formation

$$\mathrm{Nat} : \mathbf{Set}$$

Introduction

$$0 : \mathrm{Nat}$$

$$\mathrm{s} \ : \mathrm{Nat} \to \mathrm{Nat}$$

Elimination

$$\frac{B : \mathbf{Set} \quad h : B \qquad \begin{array}{c}[x : \mathrm{Nat}][y : B] \\ \vdots \\ f : B\end{array}}{\mathrm{Rec}\ h\ ([x : \mathrm{Nat}][y : B]f) \ : \ \mathrm{Nat} \to B}$$

Slide 11

$$\frac{\begin{array}{c}[x : \mathrm{Nat}] \\ \vdots \\ P : \mathbf{Prop}\end{array}}{\begin{array}{l}\mathrm{I}nd\ ([x : \mathrm{Nat}]P) : \\ P[x := 0] \to (\{n : \mathrm{Nat}\}P[x := n] \to P[x := \mathrm{s}\ n]) \to \{n : \mathrm{Nat}\}P[x := n]\end{array}}$$

Computation

$$\mathrm{Rec}\ h\ ([x : \mathrm{Nat}][y : B]f)\ 0\ = h$$
$$\mathrm{Rec}\ h\ ([x : \mathrm{Nat}][y : B]f)\ (\mathrm{s}\ n) = f[x := n, y := \mathrm{Rec}\ h\ ([x : \mathrm{Nat}][y : B]f)\ n\ ]$$

Slide 12

Examples

```
[d = Rec 0 ([m:Nat][n:Nat]s (s n))];

[p = Rec 0 ([m:Nat][n:Nat]m)];

[add = Rec ([n:Nat]n)
          ([m:Nat][f:Nat -> Nat][n:Nat]s (f n))];
```

Slide 13

Examples for Ind

```
Goal n,m:NatId (add n (s m)) (add (s n) m);
Intros n;
Refine Ind [n:Nat]m:NatId (add n (s m)) (add (s n) m);
Intros;
Refine refl;
Intros;
Refine respId s;
Refine H;
Save add_n_Sm;
```

Slide 14

```
[n:Nat]
Ind ([n'2:Nat]m:NatId (add n'2 (s m)) (add (s n'2) m))
 ([m:Nat]refl (add O (s m)))
 ([n1:Nat][H:m:NatId (add n1 (s m)) (add (s n1) m)]
      [m:Nat]respId s (H m))
 n
```

Slide 15

```
Goal n:NatExists [m:Nat]Id (add n n) m;
Intros n;
Refine Ind [n:Nat]Exists ([m:Nat]Id (add n n) m);
Refine existsI;
Refine O;
Refine refl;
Intros n H;
Refine existsE ? H;
Intros m H1;
Refine existsI;
Refine s (s m);
Refine respId s;
Refine transId;
Next +1;
Refine add_n_Sm;
Refine respId s;
Refine H1;
Save dd;
```

Slide 16

```
[n:Nat]
Ind ([n'2:Nat]Exists ([m:Nat]Id (add n'2 n'2) m))
 (existsI O (refl (add O O)))
 ([n1:Nat][H:Exists ([m:Nat]Id (add n1 n1) m)]
  existsE ([m:Nat][H1:Id (add n1 n1) m]
      existsI (s (s m))
      (respId s (transId (add_n_Sm n1 n1) (respId s H1)))) H)
 n
```

Slide 17

Axiom of Choice

Given $R : S \to T \to \textbf{Prop}$

$$\frac{\{x : S\}\text{Exists } ([y : T]Rxy)}{\text{Exists } ([f : S \to T]\{x : S\}Rx(fx))}$$

Proposition: This rule is *admissable* in the system we consider.

The proof uses the *modified realizability interpretation.*

Slide 18

Exercises

To do the exercises one has to use the file `ha.l` from the web page.

Power of 2 Define $\text{exp2} : \text{Nat} \to \text{Nat}$ s.t.

$$\begin{aligned} \text{exp2 } 0 &= \text{s } 0 \\ \text{exp2 } (\text{s } n) &= \text{add } (\text{exp2 } n)\ (\text{exp2 } n) \end{aligned}$$

Multiplication Define $\text{mult} : \text{Nat} \to \text{Nat} \to \text{Nat}$ s.t.

$$\begin{aligned} \text{mult } 0\ n &= 0 \\ \text{mult } (\text{s } m)\ n &= \text{add } n\ (\text{mult } m\ n) \end{aligned}$$

Slide 19

Divide by 2 Define div2 : Nat $\rightarrow$ Nat s.t.

$$\begin{aligned} \text{div2 } 0 &= 0 \\ \text{div2 (s 0)} &= 0 \\ \text{div2 (s (s } n)) &= \text{s (div2 } n) \end{aligned}$$

Slide 20

Divisibility Define m is divisable by n: Div : Nat $\rightarrow$ Nat $\rightarrow$ **Prop**.

Less than Define $\leq$ as Le : Nat $\rightarrow$ Nat $\rightarrow$ **Prop**.

Prime Define n is prime as Prime : Nat $\rightarrow$ **Prop**

Slide 21

Show that addition is commutative

$$\{m, n : \mathrm{Nat}\}\mathrm{Id}\ (\mathrm{add}\ m\ n)\ (\mathrm{add}\ n\ m)$$