4 Products, coproducts and exponentials

4.1 Products

The product of two sets $A, B : \mathbf{Set}$ is $A \times B$ its elements are tuples, that is given $a : A$ and $b : B$ we can form $(a, b) : A \times B$ which are the only elements of this type. We can define the projections:

$$\pi_1 : A \times B \to A$$
$$\pi_1 (a, b) = a$$
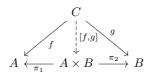$$\pi_2 : A \times B \to A$$
$$\pi_2 (a, b) = b$$

Moreover given another set $C : \mathbf{Set}$ and functions $f : A \to C$ and $g : B \to C$ we define

$$[f, g] : C \to A \times B$$
$$[f, g] \, c = (f \, c, g \, c)$$

this is the unique function that makes the following diagram commute

The uniqueness is indicated by the dashed arrow.

By unique we mean that for any other function $h : C \to A \times B$ which makes this diagram commute we have that $h = [f, g]$. Using extensionality it is sufficient to show that $h \, x = [f, g] \, x$:

$$
\begin{aligned}
[f, g] \, x &= (f \, x, g \, x) \\
&= ((\pi_1 \circ h) \, x, \pi_2 \circ h) \, x) \\
&= (\pi_1 \, (h \, x), \pi_2 \, (h \, x)) \\
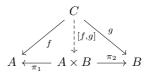&= (\lambda z.(\pi_1 \, z, \pi_2 \, z)) \circ h
\end{aligned}
$$

Using the fact that all elements of $A \times B$ are tuples it is easy to see that $\lambda z.(\pi_1 \, z, \pi_2 \, z)) = \lambda z.z = \mathrm{id}$ and hence

$$
\begin{aligned}
(\lambda z.(\pi_1 \, z, \pi_2 \, z)) \circ h &= \mathrm{id} \circ h \\
&= h
\end{aligned}
$$

Moreover, $[\_,\_]$ is natural, this means that $[f,g] \circ h = [f \circ h, g \circ h]$:

$$
\begin{aligned}
([f,g] \circ h)\,x &= [f,g]\,(h\,x)\\
&= (f\,(h\,x), g\,(h\,x))\\
&= ((f \circ h)\,x), (g \circ h)\,x))\\
&= [f \circ h, g \circ h]\,x
\end{aligned}
$$

We abstract from products in **Set** and say that in any category $\underline{\mathbf{C}}$ a product $A \times B$ of two objects A, B is given by two morphisms $\pi_1 : \underline{\mathbf{C}}(A \times B, A)$ and $\pi_2 : \underline{\mathbf{C}}(A \times B, B)$ and for any pair of morphisms $f : \underline{\mathbf{C}}(C, A)$ and $g : \underline{\mathbf{C}}(C, B)$ a unique morphism $[f,g] : \underline{\mathbf{C}}(C, A \times B)$ that makes the following diagram commute:

Moreover this assignment is natural, i.e. $[f,g] \circ h = [f \circ h, g \circ h]$.

Question 28 *Has the category $\underline{\omega}$, which corresponds to $\leq$ on natural numbers, got products? In general what are products for a preorder?*

As for terminal and initial objects we can show that products are unique up to isomorphism. This also implies that $A \times B \cong B \times A$ because it is easy to see that $B \times A$ is a product of A and B.

We have only defined binary products, what about ternary products? There are at least two ways to derive them from binary propducts $A \times (B \times C)$ and $(A \times B) \times C$.

Exercise 29 *Show that $A \times (B \times C) \cong (A \times B) \times C$.*

This suggests that we can indeed derive all non-empty products from binary products. The only things which is missing is the *empty product*, which corresponds to the terminal object. Hence we say that a category has finite products if it has a terminal object and binary products. Note that this also means that we chose a particular terminal object 1 and for any pair of object A and B another object $A \times B$.

4.2 Coproducts

It is easy to say what are coproducts: they are simply products in the opposite category. However, it is useful to develop some intuition by looking at coproducts [12] in **Set** first: Given $A, B :$ **Set** we define $A + B :$ **Set** whose elements

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[12]They are also called *disjoint union* due to the way they are defined in set theory. However, since unions don't make sense in type theory this isn't a very good name. A better alternative to coproducts is *sums*.

are $\text{inj}_1\, a : A + B$ for $a : A$ and $\text{inj}_2\, b : A + B$ for $b : B$ and these are the only elements. Now given $f : A \to C$ and $g : B \to C$ we define $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] : A + B \to C$ which allows us to do case analysis:

$$\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] (\text{inj}_1\, a) = f\, a$$
$$\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] (\text{inj}_2\, b) = g\, b$$

This is the unique function which makes the following diagram commute:

Moreover this assignment is natural, i.e. $h \circ \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} h \circ f \\ h \circ g \end{smallmatrix}\right]$.

Exercise 30 *Verify unicity and naturality of $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$.*

This leads directly to the definition of binary coproducts in a category $\underline{\mathbf{C}}$, a coproduct $A + B$ of A and B is given by injections: $\text{inj}_1 : \underline{\mathbf{C}}(A, A + B)$ for and $\text{inj}_2\, b : \underline{\mathbf{C}}(B, A + B)$ and for any pair of morphisms $f : \underline{\mathbf{C}}(A, C)$ and $g : \underline{\mathbf{C}}(B, C)$ a unique morphism $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] : \underline{\mathbf{C}}(A + B, C)$ that makes the following diagram commute:

Moreover this assignment is natural, i.e. $h \circ \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} h \circ f \\ h \circ g \end{smallmatrix}\right]$.

Question 31 *Has the category $\underline{\omega}$, which corresponds to $\le$ on natural numbers, got coproducts? In general what are coproducts for a preorder?*

As for produts we say that a category has all finite coproducts if it has an initial object and all binary coproducts.

In **Set** and actually in many categories products and coproducts are very different, but this is not always the case. We have already seen that in **Mon** ths initial object and the terminal object coincide. For the following consider the category of commutative monoids **CMon** which is like monoids but with the additional property that $x * y = y * x$ for any object $(A, e, \_ * \_)$. Given two commutative monoids $(A, e, \_ * \_), (B, z, \_ + \_)$ we define $A \oplus B$ as $(A \times B, (e, z), \_ \bullet \_)$ where $(a, b) \bullet (a', b') = (a * a', b + b')$. It is easy to see that this is a commutative monoid again.

Exercise 32 *Show that $A \oplus B$ is both a product and a coproduct of A and B.*

Question 33 *What happens if we give up commutativity, i.e. if we consider* **Mon** *instead.*

Products and coproducts can also be defined as an adjunction: Given a category $\underline{\mathbf{C}}$ we write $\underline{\mathbf{C^2}} = \underline{\mathbf{C}} \times \underline{\mathbf{C}}$ [13] whose objects are pairs of objects and whose morphsims are pairs of morphisms in $\underline{\mathbf{C}}$. There is a functor $\Delta : \underline{\mathbf{C}} \to \underline{\mathbf{C^2}}$ (called the diagonal functor) which just copies, that is $\Delta A = (A, A)$ and the same on morphisms. Now if $\underline{\mathbf{C}}$ has binary products and coproducts resepctively we also have $\_\times\_,\_+\_ : \underline{\mathbf{C^2}} \to \underline{\mathbf{C}}$

Exercise 34 *Define the morphism parts of these functors using only the properties of products / coproducts.*

Moreover, they are left and right adjoint to Δ:

Exercise 35 *Verify these adjunctions, i.e. show that*

$$\underline{\mathbf{C}}(X, Z) \times \underline{\mathbf{C}}(Y, Z) \cong \underline{\mathbf{C}}(X + Y, Z)$$
$$\underline{\mathbf{C}}(Z, X \times Y) \cong \underline{\mathbf{C}}(Z, X) \times \underline{\mathbf{C}}(Z, Y)$$

are natural isomorphisms.

4.3 Exponentials

Next we look at the set of functions which is called an *exponential* in category theory. One of the basic properties of functions in functional programming is currying, which is the following isomorphism:

where

$$\phi_{A,B,C} : (A \times B \to C) \to (A \to (B \to C))$$
$$\phi\ f = \lambda x.\lambda y.f\,(x, y)$$
$$\psi_{A,B,C} : (A \to (B \to C)) \to (A \times B \to C)$$
$$\psi\ g = \lambda p.g\,(\pi_1\,p)\,(\pi_2\,p)$$

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[13]Yes, this is a product in the category of categories which is not a category in our sense. We already develop a taste for higher categories. . .

Exercise 36 *Verify that ϕ and ψ are inverse to each other.*

If we fix B this looks like an adjunction, namely the functors $\_ \times B, B \to \_ :$ **Set** $\to$ **Set** form an adjunction. All that is left is to check naturality:

Exercise 37 *Derive the morphism part of $\_ \times B$ and $B \to \_$ and check that $\phi_{A,B,C}$ is natural in A, C, that is given $f : A \to A'$ and $g : C \to C'$*

$$
\begin{array}{ccc}
A \times B \to C & \xrightarrow{\phi_{A,B,C}} & A \to (B \to C) \\
\downarrow{\scriptstyle \lambda h.g \circ h \circ (f \times B)} & & \downarrow{\scriptstyle \lambda k.(B \to g) \circ k \circ f} \\
A' \times B \to C' & \xrightarrow[\phi_{A',B,C'}]{} & A' \to (B \to C')
\end{array}
$$

This directly leads to the definition of *exponentials (aka function objects)* for a category $\underline{\mathbf{C}}$ with products, we say that for a fixed object $B : |\underline{\mathbf{C}}|$ an exponential $B \to \_$ or $(\_)^B$ is the right adjoint to $\_ \times B$. We can compare $\_ \to \_$ with homsets $\underline{\mathbf{C}}(\_,\_)$ both take two objects as input but the first one returns an object while the latter returns a set. This is why exponentials are also called internal homs. Only on **Set** the two are actually the same.

By definition $B \to \_$ is a covariant functor $\underline{\mathbf{C}} \to \underline{\mathbf{C}}$ but exponentials also give rise to a contravariant functor:

Exercise 38 *Show that $\_ \to C$ is a contravariant functor, i.e.*

$$\_ \to C : \underline{\mathbf{C}}^{\mathrm{op}} \to \underline{\mathbf{C}}.$$

A category which has all finite products and exponentials is called a *a cartesian closed category* (CCC) if it is also has coproducts it is a *Bicartesian closed category (BiCCC)*. As we have already observed **Set** is cartesian closed and has finite coproducts hence it is a bicartesian closed category.

The following example of a bicartesian closed category is a good justification for the notation: the category of finite sets $\underline{\mathrm{Fin}}$: its objects are natural numbers, and its homsets are defined as $\underline{\mathrm{Fin}}(m,n) = \mathrm{Fin}\, m \to \mathrm{Fin}\, n$ where $\mathrm{Fin}\, k = \{i : \mathbb{N} \mid i < k\}$, i.e. $\mathrm{Fin}\, k$ has exactly k elements. $\underline{\mathrm{Fin}}$ has (given $m, n : |\underline{\mathrm{Fin}}| = \mathbb{N}$):

an initial object	0
coproducts	$m + n$
a terminal object	1
products	$m \times n$
exponentials	$m \to n = n^m$

Exercise 39 *Verify that these definitions indeed have the required properties.*

Another standard example for a bicartesian closed category is the category of propositions **Prop**: its objects are propositions and its homsets are given by implication (which correspond to functions): $\underline{\mathbf{Prop}}(P, Q) = P \to Q$. This category is a preorder (because all its homs are propositions) and it has (given $P, Q : \mathbf{Prop}$) :

an initial object	False
coproducts	$P \vee Q$
a terminal object	True
products	$P \wedge Q$
exponentials	$P \rightarrow Q$

Exercise 40 *Verify that these definitions indeed have the required properties.*

A bicartesian closed category which is a preorder is called a *Heyting algebra*, it is a model for intuitionistic propositional logic.

In every bicartesian closed category products and coproducts distribute as in ordinary algebra. That means we have the following isomorphisms

$$0 \times A \cong 0$$
$$(B + C) \times A \cong B \times A + C \times A$$

We don't need the symmetric cases since we alreay know that commutativity of products ($A \times B \cong B \times A$) and coproducts ($A + B \cong B + A$) are isomorphisms.

The easiest way to show this is to use the corollary from the Yoneda lemma: to show an isomorphism it is enough to show that the homsets out of (or into) these objects are naturally isomorphic. Let's do the 2nd isomorphism:

$$\begin{aligned}
\underline{\mathbf{C}}((B + C) \times A, X) &\cong \underline{\mathbf{C}}(B + C, A \rightarrow X) \\
&\cong \underline{\mathbf{C}}(B, A \rightarrow X) \times \underline{\mathbf{C}}(C, A \rightarrow X) \\
&\cong \underline{\mathbf{C}}(B \times A, X) \times \underline{\mathbf{C}}(C \times A, X) \\
&\cong \underline{\mathbf{C}}(B \times A + C \times A, X)
\end{aligned}$$

and hence $(B + C) \times A \cong B \times A + C \times A$. In each line we only used the adjunctions defining $\times$, $+$ and $\rightarrow$.

Exercise 41 *Prove the first isomorphism $0 \times A \cong 0$ using the same idea. Note that $\underline{\mathbf{C}}(0, A) \cong 1$*

Not every category with products and coproducts is distributive. A counterexample is **Mon** where the terminal object and the initial object agree, i.e. $0 \cong 1$. hence we can just use the first isomorphism to show that every object is isomorphic to the initial object (and the terminal):

$$\begin{aligned}
A &\cong 1 \times A \\
&\cong 0 \times A \\
&\cong 0
\end{aligned}$$

But this is certainly not true in **Mon**, hence we can conclude that **Mon** is not cartesian closed.

Exercise 42 *Is $\underline{\omega}$ (bi)cartesian closed? What about $\underline{\omega}^{\mathrm{op}}$?*

An important example for a bicartesian closed category is the category of presheaves of a given category $\mathbf{PSh}\,\underline{\mathbf{C}}$. Its objects are functors $F : \underline{\mathbf{C}}^{\mathrm{op}} \to \mathbf{Set}$ and its morphisms are natural transformations. Products (and coproducts) are calculated *pointwise*: That is given $F, G : \underline{\mathbf{C}}^{\mathrm{op}} \to \mathbf{Set}$

$$(F \times G)\,A = F\,A \times G\,A$$
$$(F + G)\,A = F\,A + G\,A$$

Exercise 43 *Verify that these are indeed products and coproducts in* $\mathbf{PSh}\,\underline{\mathbf{C}}$.

But what about exponentials? We cannot have $(F \to G)\,A = F\,A \to G\,A$ because this wouldn't be a contravariant functor since F appears on the left hand side of the exponential. The construction of the exponential is actually a nice application of the Yoneda lemma: If we assume that the exponential exists we have then

$$
\begin{aligned}
(F \to G)\,A &= \int_{X:\underline{\mathbf{C}}} C(X, A) \to (F \to G)\,X \\
&= \mathbf{PSh}\,\underline{\mathbf{C}}(C(\_, A), F \to G) \\
&= \mathbf{PSh}\,\underline{\mathbf{C}}(C(\_, A) \times F, G) \\
&= \int_{X:\underline{\mathbf{C}}} C(X, A) \times F\,X \to G\,X \\
&= \int_{X:\underline{\mathbf{C}}} C(X, A) \to F\,X \to G\,X
\end{aligned}
$$

So the last line is a candidate for the exponential but in this case we do need to think about size. If the type of objects is not small (for example it could be $\mathbf{Set}$) then the set of natural transformations isn't small either. However, if we assume that $|\underline{\mathbf{C}}|$ is small we can just define

$$(F \to G)\,A = \int_{X:\underline{\mathbf{C}}} C(X, A) \to F\,X \to G\,X$$

This shows that the category or presheaves over a small category (i.e. a category with a small type of objects) is cartesian closed. However, it does for example not imply that the category of presheaves over $\mathbf{Set}$, i.e. functors $\underline{\mathbf{Set}}^{\mathrm{op}} \to \underline{\mathbf{Set}}$ is cartesian closed.

Given a preorder, that is $|R| : \mathbf{Set}$ and $\_R\_ : |R| \to |R| \to \mathbf{Prop}$ that is reflexive and transitive, a propositional presheaf $P : \underline{\mathbf{R}}^{\mathrm{op}} \to \mathbf{Prop}$ is just a predicate $P : |R| \to \mathbf{Prop}$ that is inverse monotone wrt R, i.e. if $P\,x$ and $y\,R\,x$ then $P\,y$. Morphisms are given by pointwise inclusion

$$(\underline{\mathbf{PSh}\,\mathbf{R}})(P, Q) = \forall x : |R|.P\,x \to Q\,x.$$

This is a bicartesian closed category the constructions are simplified versions of the ones for set valued presehaves, in particular

$$(P \to Q)\,x = \forall y : |R|.y\,R\,x \to P\,y \to Q\,y$$

These categories are called *Kripke models* and the fact that they are bicartesian closed corresponds to the fact that they model intuitionistic propositional logic.

Exercise 44 *Use Kripke models to show that the law of the excluded middle $P \lor \neg P$ is not derivable in intuitionistic logic.*

You may have noticed that bicartesian closed categories are a bit asymmetric. They have a right adjoint to $\_ \times B$ but what about a left adjoint to $B + \_$? Let's denote such an object as $\_ - B$ (minus); it should satisfy the following natural isomorphism:

$$\underline{\mathbf{C}}(A, B + C) \cong \underline{\mathbf{C}}(A - B, C)$$

natural in C.

An example for such a *symmetric cartesian closed category* is the category of booleans which is similar to the category of propositions, but we use booleans instead of propositions. In particular, the type of objects is Bool and the homsets are given as $\underline{\text{Bool}}(p, q) = \mathrm{T}\, p \to \mathrm{T}\, q$ where $\mathrm{T} : \text{Bool} \to \mathbf{Prop}$ is defined as $\mathrm{T}\, b = (b = \text{true})$. We can define products, coproducts and exponentials as for **Prop**, i.e. by using the usual truthtable semantics. Subtraction can be defines as $p - q = p \land \neg q$.

Exercise 45 *Check that $\underline{\text{Bool}}$ is a symmetric cartesian closed category.*

You may notice that this example is a preorder. This is no accident because all symmetric cartesian closed categories are preorders. This is because in such a category also the dual form of distributivity hold:

$$1 + A \cong 1$$
$$(B \times C) + A \cong (B + A) \times (C + A)$$

The first one is already suspicious because it implies in particular that $1+1 = 1$, which basically says that true = false in Bool. Moreover:

Exercise 46 *In a bicartesian closed category with $1 = 1 + 1$ all morphisms are equal (i.e. it is a preorder).*

You may wonder wether symmetry already implies excluded middle $P \lor \neg P$, which certainly holds in $\underline{\text{Bool}}$. The answer is no, because we can extend Kripke semantics to the symmetric logic (using boolean valued presheaves over finite preorders) and uses this to show that excluded middle isn't implied.

Exercise 47 *Show that the category $\mathbf{1} + \omega$ is a symmetric cartesian closed category. It's definition is similar to $\underline{\omega}$ its objects are the natural numbers and a top element ω such that $i \leq \omega$ for all elements.*