Monads and More: Part 3

Tarmo Uustalu, Tallinn

Nottingham, 14–18 May 2007

Arrows (Hughes)

- Arrows are a generalization of strong monads on symmetric monoidal categories (in their Kleisli triple form).
- An *arrow* on a symmetric monoidal category $(\mathcal{C}, I, \otimes)$ is given by
  - an object mapping $R : |\mathcal{C}| \times |\mathcal{C}| \to |\mathbf{Set}|$,
  - for any objects A, B of $\mathcal{C}$, a map $\mathrm{arr} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to R(A, B)$ of $\mathbf{Set}$,
  - for any objects A, B, C of $\mathcal{C}$, a map $\lll: R(A, B) \times R(B, C) \to R(A, C)$ of $\mathbf{Set}$,
  - for any objects A, B, C of $\mathcal{C}$, a map $\mathrm{second}_C : R(A, B) \to R(C \otimes A, C \otimes B)$ of $\mathbf{Set}$

  satisfying the conditions on the next slide.

- (ctd. from the previous slide)
  - if $k \in R(A, B)$, then $\mathrm{arr}\,\mathrm{id}_B \lll k = k$,
  - if $k \in R(A, B)$, then $k \lll \mathrm{arr}\,\mathrm{id}_A = k$,
  - if $k \in R(A, B)$, $\ell \in R(B, C)$, $m \in R(C, D)$, then $(m \lll \ell) \lll k = m \lll (\ell \lll k)$,
  - if $f : A \to B$, $g : B \to C$, then $\mathrm{arr}\,(g \circ f) = \mathrm{arr}\,g \lll \mathrm{arr}\; f$,
  - if $f : A \to B$, then $\mathrm{second}_C\,(\mathrm{arr}\,f) = \mathrm{arr}(\mathrm{id}_C \times f)$,
  - if $k \in R(A, B)$, $\ell \in R(B, C)$, $\mathrm{second}_D\,(\ell \lll k) = \mathrm{second}_D\,\ell \lll \mathrm{second}_D\,k$,
  - if $k \in R(A, B)$, $f : C \to D$, then $\mathrm{arr}\,(f \times \mathrm{id}_B) \lll \mathrm{second}_C\,k = \mathrm{second}_D\,k \lll \mathrm{arr}\,(f \times \mathrm{id}_A)$,
  - if $k \in R(A, B)$, $k \lll \mathrm{arr}, \mathrm{ul}_A = \mathrm{ul}_B \lll \mathrm{second}_I k$,
  - if $k \in R(A, B)$, $\mathrm{second}_C\,(\mathrm{second}_D\,k) \lll \mathrm{a}_{C,D,A} = \mathrm{a}_{C,D,B} \lll \mathrm{second}_{C \otimes D} k$.

Examples

- Arrows from monoidal functors.
  - $R(A, B) =_{\mathrm{df}} \mathrm{Hom}_{\mathcal{C}}(FA, FB)$ where F is a monoidal endofunctor on $\mathcal{C}$ (i.e., there is a natural iso $\mathsf{m}_{A,B} : FA \otimes FB \to F(A \otimes B)$),
  - if $f : A \to B$, then $\mathsf{arr}\, f = Ff : FA \to FB$,
  - if $k : FA \to FB$, $\ell : FB \to FC$, then $\ell \lll k =_{\mathrm{df}} A \xrightarrow{Fk} FB \xrightarrow{F\ell} FC$,
  - if $k : FA \to FB$, then $\mathsf{second}\, k =_{\mathrm{df}} F(C \otimes A) \xrightarrow{\mathsf{m}^{-1}} FC \otimes FA \xrightarrow{\mathrm{id} \otimes Fk} FC \otimes FB \xrightarrow{\mathsf{m}} F(C \otimes B)$.

- Kleisli maps of strong monads.
    - $R(A, B) =_{\mathrm{df}} \mathrm{Hom}_{\mathcal{C}}(A, TB)$ where T is a strong monad,
    - if $f : A \to B$, then $\mathrm{arr}\, f = Jf : A \to TB$ where J is the Kleisli inclusion of T,
    - if $k : A \to TB$, $\ell : B \to TC$, then $\ell \lll k =_{\mathrm{df}} A \xrightarrow{k} TB \xrightarrow{\ell^\star} TC$,
    - if $k : A \to TB$, then $\mathrm{second}\, k =_{\mathrm{df}} C \otimes A \xrightarrow{\mathrm{id} \otimes k} C \otimes TB \xrightarrow{\mathrm{sr}} T(C \otimes B)$.
- CoKleisli maps of comonads on Cartesian categories.
    - $R(A, B) =_{\mathrm{df}} \mathrm{Hom}_{\mathcal{C}}(DA, B)$ where D is a comonad on $\mathcal{C}$,
    - if $f : A \to B$, then $\mathrm{arr}\, f = Jf : DA \to B$ where J is the coKleisli inclusion of D,
    - if $k : DA \to B$, $\ell : DB \to C$, then $\ell \lll k =_{\mathrm{df}} DA \xrightarrow{k^\dagger} DB \xrightarrow{\ell} C$,
    - if $k : DA \to B$, then $\mathrm{second}\, k =_{\mathrm{df}} D(C \times A) \xrightarrow{\langle D\mathrm{fst}, D\mathrm{snd} \rangle} DC \times DA \xrightarrow{\varepsilon \times k} C \times B$.

- Output once more:
  - $R(A, B) =_{\mathrm{df}} E \times \mathrm{Hom}_{\mathcal{C}}(A, B)$ where (E, e, m) is a monoid in **Set**,
  - if $f : A \to B$, then $\mathrm{arr}\, f = (e, f) : E \times \mathrm{Hom}_{\mathcal{C}}(A, B)$,
  - if $(x, f) : E \times \mathrm{Hom}_{\mathcal{C}}(A, B)$, $(y, g) : E \times \mathrm{Hom}_{\mathcal{C}}(B, C)$, then
    $(y, g) \lll (x, f) =_{\mathrm{df}} (m(x, y), g \circ f) \in E \times \mathrm{Hom}_{\mathcal{C}}(A, C)$,
  - if $(x, f) : E \times \mathrm{Hom}_{\mathcal{C}}(A, B)$, then
    $\mathrm{second}\, (x, f) =_{\mathrm{df}} (x, C \otimes f) \in E \times \mathrm{Hom}_{\mathcal{C}}(C \otimes A, C \otimes B)$.

Arrows in the monoid form (Jacobs, Heunen, Hasuo)

- An alternative definition mimicks the definition of monads in the standard, i.e., monoid form.
- An *arrow* on a symmetric monoidal category $(\mathcal{C}, I, \otimes)$ is a strong monoid in the category of endoprofunctors on $(\mathcal{C}, I, \otimes)$.
- A *profunctor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$.
  The identity profunctor on $\mathcal{C}$ is
  $\mathrm{Id} =_{\mathrm{df}} \mathrm{Hom}_{\mathcal{C}} : C^{\mathrm{op}} \times C \to \mathbf{Set}$.
  The composition of profunctors $R : \mathcal{C} \to \mathcal{D}$ and $S : \mathcal{D} \to \mathcal{E}$ is $SR(A, C) =_{\mathrm{df}} \int^{B} R(A, B) \otimes S(B, C)$.

- Accordingly, the data of an arrow are the following.
  - The carrier of an arrow is a profunctor R from $\mathcal{C}$ to $\mathcal{C}$, i.e., a functor $R : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.
  - The unit is a natural transformation from Id to R, i.e., a family of maps $\mathsf{arr}_{A,B} : \mathsf{Hom}_{\mathcal{C}}(A, B) \to R(A, B)$ natural in A, B.

    The multiplication is a nat. transf. from RR to E, i.e., a family of maps $\lll_{A,B,C}: R(A, B) \times R(B, C) \to R(A, C)$ natural in A, C and dinatural in B.

    The strength is a family of $\mathsf{second}_{A,B,C} :: R(A, B) \to R(C \otimes A, C \otimes B)$ natural in A, B and dinatural in C.

Symmetric premonoidal categories (Power, Robinson)

- Intuitively, a symmetric premonoidal category is the same as a symmetric monoidal category, except that the tensor is not necessarily a bifunctor, it must only be functorial in each argument separately.
- More officially: A *symmetric premonoidal category* is given by
  - a category $\mathcal{K}$,
  - an object I of $\mathcal{K}$,
  - for any object C, a functor $C \rtimes - : \mathcal{K} \to \mathcal{K}$,
  - natural isomorphisms a, ul, ur, c satisfying the laws of a symmetric monoidal category and have all their components central (see further).

- Symmetry yields a symmetric functor $- \ltimes C : \mathcal{K} \to K$ where $A \ltimes B = A \rtimes B$.
- A morphism $f : A \to B$ is called *central* if, for any $g : C \to D$, both

$$
\begin{aligned}
(f \ltimes D) \circ (A \rtimes g) &= (B \rtimes g) \circ (f \ltimes C) \\
(D \rtimes f) \circ (g \ltimes A) &= (g \ltimes B) \circ (C \rtimes f)
\end{aligned}
$$

Freyd categories

- A Freyd category on a symmetric monoidal category $\mathcal{C}$ is given by
  - a symmetric premonoidal category $(\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}})$
  - together with an identity-on-objects inclusion functor $J : \mathcal{C} \rightarrow \mathcal{K}$ that preserves centrality and strictly preserves its the $(I, \otimes)$ structure as premonoidal (meaning that $I^{\mathcal{K}} = I$, $A \otimes^{\mathcal{K}} B = A \otimes B$).

Freyd categories vs. arrows (Jacobs, Heunen, Hasuo)

- Freyd categories are in a bijection with arrows.
- For an arrow R on a symmetric monoidal category $(\mathcal{C}, I, \otimes)$, the Freyd category $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$ is defined by
  - an object is an object of $\mathcal{C}$,
  - a map from A to B is an element of $R(A, B)$,
  - $\mathsf{id}^{\mathcal{K}}_A =_{\mathrm{df}} \mathsf{arr}\,\mathsf{id}_A$,
  - if $k : A \to^{\mathcal{K}} B$, $\ell : B \to^{\mathcal{K}} C$, then $\ell \circ^{\mathcal{K}} k =_{\mathrm{df}} \ell \lll k$,
  - $I^{\mathcal{K}} = I$, $A \otimes^{\mathcal{K}} B =_{\mathrm{df}} A \otimes B$,
  - if $k : A \to^{\mathcal{K}} B$, then $C \otimes^{\mathcal{K}} k =_{\mathrm{df}} \mathsf{second}\,k$,
  - if $f : A \to B$, then $Jf =_{\mathrm{df}} \mathsf{arr}\,f$.

- Given a Freyd category $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$ on $\mathcal{C}$, the corresponding arrow R is defined by
  - $R(A, B) =_{\mathrm{df}} \mathsf{Hom}_{\mathcal{K}}(A, B)$,
  - if $f : A' \to A$, $g : B \to B'$, $k \in \mathsf{Hom}_{\mathcal{K}}(A, B)$, then $R(f, g)\, k =_{\mathrm{df}} \mathsf{arr}\, g \lll k \lll \mathsf{arr}\, f$,
  - if $f : A \to B$, then $\mathsf{arr}\, f =_{\mathrm{df}} Jf \in \mathsf{Hom}_{\mathcal{K}}(A, B)$,
  - if $k \in \mathsf{Hom}_{\mathcal{K}}(A, B)$, $\ell \in \mathsf{Hom}_{\mathcal{K}}(B, C)$, then $\ell \lll k =_{\mathrm{df}} \ell \circ^{\mathcal{K}} k \in \mathsf{Hom}_{\mathcal{K}}(A, C)$,
  - if $k \in \mathsf{Hom}_{\mathcal{K}}(A, B)$, then $\mathsf{second}\, k =_{\mathrm{df}} C \otimes^{\mathcal{K}} k \in \mathsf{Hom}_{K}(C \otimes A, C \otimes B)$.

When is Freyd Kleisli? (Power)

- Given a Freyd category $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$ on a symmetric monoidal category $(\mathcal{C}, I, \otimes)$, when is it the Kleisli category of a strong monad?
- A simple condition is in terms of Kleisli exponents.
- Suppose $J(- \otimes A) : \mathcal{C} \to \mathcal{K}$ has a right adjoint $A \Rightarrow^{\mathcal{K}} -$. In this case we say the Freyd category is *closed*. Then also $TB =_{\mathrm{df}} I \Rightarrow^{\mathcal{K}} B$ is a strong monad with Kleisli exponents and $((\mathcal{K}, I^{\mathcal{K}}, \otimes^{\mathcal{K}}), J)$ is its Kleisli category.