Exercises for Naive Type Theory
Part 1

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1. Given

   $$\begin{aligned}
   &f : \mathbb{N} \to \mathbb{N} \\
   &f :\equiv \lambda x.x + x \\
   &g : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N} \\
   &g :\equiv \lambda h.\lambda n.h\,(h\,n) + n
   \end{aligned}$$

   Evaluate $g\,f\,(f\,1)$ justifying each step and underline the affected subterm.

2. For any types A, B, C we define identity and composition as follows:

   $$\begin{aligned}
   \mathrm{id}_A &: A \to A \\
   \mathrm{id}_A &:\equiv \lambda x.x \\
   \mathrm{cmp}_{A,B,C} &: (B \to C) \to (A \to B) \to A \to C \\
   \mathrm{cmp}_{A,B,C} &:\equiv \lambda f, g.\lambda x.f\,(g\,x)
   \end{aligned}$$

   Writing $\mathrm{cmp}_{A,B,C}\,f\,g$ as $f \circ g$ show that this data forms a category by establishing

   (a) $f \circ \mathrm{id} \equiv f$

   (b) $\mathrm{id} \circ f \equiv f$

   (c) $(f \circ g) \circ h \equiv f \circ (g \circ h)$

   for appropriate f, g, h. You can use η-equality - where do we need it?

3. For any types A, B, C find an element of

   $$((A \to C) \to C) \to (A \to B) \to ((B \to C) \to C)$$