Exercises for Naive Type Theory
Part 3

Thorsten Altenkirch

April 14, 2016

1. Given a type A and $n : \mathbb{N}$ we define the type $\mathrm{Vec}\,A\,n$ as given by the constructors:

$$[] : \mathrm{Vec}\,A\,0$$
$$- :: - : \Pi_{n:\mathbb{N}} A \to \mathrm{Vec}\,A\,n \to \mathrm{Vec}\,A\,(1+n)$$

   We define lists over A the following way:

$$\mathrm{List}\,A :\equiv \Sigma n : \mathbb{N}.\mathrm{Vec}\,A\,n$$

   Given this derive the following operations on lists:

$$\mathrm{nil} : \mathrm{List}\,A$$
$$\mathrm{cons} : A \to \mathrm{List}\,A \to \mathrm{List}\,A$$
$$\mathrm{fold} : X \to (A \to X \to X) \to \mathrm{List}\,A \to X$$

   using general recursive definitions by pattern matching.

2. Given $n : \mathbb{N}$ we define the type $\mathrm{Fin}\,n$ as given by the following constructors

$$0 : \Pi_{n:\mathbb{N}} \mathrm{Fin}\,1+n$$
$$\mathrm{suc} : \Pi_{n:\mathbb{N}} \mathrm{Fin}\,n \to \mathrm{Fin}\,(1+n)$$

   Define the following functions

$$\mathrm{max} : \Pi_{n:\mathbb{N}} \mathrm{Fin}(1+n)$$
$$\mathrm{emb} : \Pi_{n:\mathbb{N}} \mathrm{Fin}\,n \to \mathrm{Fin}\,(1+n)$$
$$\mathrm{inv} : \Pi_{n:\mathbb{N}} \mathrm{Fin}\,n \to \mathrm{Fin}\,(1+n)$$

   where max computes the maximal element in a non-empty finite type; emb embedds $\mathrm{Fin}\,n$ into $\mathrm{Fin}\,(1+n)$ without changing the value, e.g. $\mathrm{emb}\,3_5 \equiv 3_6$. The function inv inverts the order of elements in a finite type in particular it maps 0 to the maximal element and the maximal element to 0.

3. Using propositions as types, can you prove the translation of the axiom of choice? Let A, B be type and R be a relation between A and B the axiom of choice is:

   $(\forall x : A.\exists y : B.R\,x\,y) \Rightarrow (\exists f : A \to B.\forall x : A.R\,x\,(f\,x))$

4. Using propositions as types try to prove the de Morgan laws of predicate logic:

   $\neg(\forall x : A.P\,x) \Leftrightarrow \exists x : A.\neg(P\,x)$
   $\neg(\exists x : A.P\,x) \Leftrightarrow \forall x : A.\neg(P\,x)$