Exercises for Naive Type Theory
Part 4

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1. Given the definition of $\mathrm{T}ree : \mathbf{Type}$ by the following constructor:

   $\mathrm{mkTree} : \Pi I : \mathbf{Type}.(I \to \mathrm{Tree}) \to \mathrm{Tree}$

   Implement the following axioms of set theory as operations on trees:

   (a) Empty set axiom

   $\exists x.\forall y.\neg(y \in x)$

   (b) Axiom of pairing

   $\forall x, y.\exists z.x \in z \wedge y \in z$

   (c) Axiom of union

   $\forall x.\exists y.(\forall z.z \in x.\forall w.w \in z \Rightarrow w \in y)$

   (d) Axiom of infinity

   $\exists x.\emptyset \in x \wedge \forall y.y \in x \Rightarrow y \cup \{y\} \in x$

   (e) Powerset axiom

   $\forall x.\exists y.\forall z.z \subseteq x \Rightarrow z \in y$

2. Define the function

   $\min : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$

   which computes the minimum of two natural numbers first using pattern matching style and then only using

   $\mathrm{rec}_A : A \to (\mathbb{N} \to A \to A) \to \mathbb{N} \to A$

$$\mathrm{rec}\, z\, s\, 0 :\equiv z$$
$$\mathrm{rec}\, z\, s\, (\mathrm{suc}\, m) :\equiv s\, n\, (\mathrm{rec}\, z\, s\, m)$$

Prove that

$\forall x, y : \mathbb{N}. \min x\, y = \min y\, x$

using propositions as types. First give a proof using pattern matching style, then give a proof only using the eliminator:

$\mathrm{elim} : \Pi A : \mathbb{N} \to \mathbf{Type}. A\, 0 \to (\Pi n : \mathbb{N}. A\, n \to A\, (\mathrm{suc}\, n)) \to \Pi n : \mathbb{N}. A\, n$

$$\mathrm{elim}\, A\, z\, s\, 0 :\equiv z$$
$$\mathrm{elim}\, A\, z\, s\, (\mathrm{suc}\, n) :\equiv s\, n\, (\mathrm{elim}\, A\, z\, s\, n)$$

3. We define the dependent type

   $\mathrm{In} : A \to \mathrm{List}\, A \to \mathbf{Type}$

   as given by the constructors:

   $\mathrm{here} : \Pi a : A. \Pi l : \mathrm{List}\, A. \mathrm{In}\, a\, (a :: l)$
   $\mathrm{later} : \Pi a, b : A. \Pi l : \mathrm{List}\, A. \mathrm{In}\, a\, l \to \mathrm{In}\, a\, (b :: l)$

   Derive the recursor (non dependent eliminator) and the dependent eliminator for this In.