1st Note for *A Taste of Proof Theory*

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Formulas of propositional logic are given by trees of the form

$$A :: P \mid \top \mid \bot \mid A \Rightarrow A \mid A \wedge A \mid A \vee A$$

where P is the set of atomic propositions. When writing formula trees we associate $\Rightarrow$ to the right, and assume that $\vee$ and $\wedge$ bind stronger than $\Rightarrow$. We use the abbrevations:

$$\begin{array}{rcl} \neg A & \equiv & A \Rightarrow \bot \\ A \Leftrightarrow B & \equiv & (A \Rightarrow B) \wedge (B \Rightarrow A) \end{array}$$

Here $\neg$ binds stronger than any other operator and $\Leftrightarrow$ behaves like $\Rightarrow$.

We present the rules without proof terms here.

1 Hilbert style (H)

Here I present only the assumption-free calculus with sequents of the style $\vdash_{\mathrm{H}} A$. To obtain (H) with assumptions one has to define contexts as explained below, add the rule (Hyp) and relativize all axioms and the rule (MP) wrt any given context.

1.1 Implication

$$\frac{}{\vdash_{\mathrm{H}} A \Rightarrow B \Rightarrow A}\,\mathrm{K} \qquad \frac{}{\vdash_{\mathrm{H}} (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C}\,\mathrm{S}$$

$$\frac{\vdash_{\mathrm{H}} A \Rightarrow B \qquad A}{\vdash_{\mathrm{H}} B}\,\mathrm{MP}$$

1.2 Propositional logic

We add the following families of axioms:

$$\begin{array}{l}
\vdash_{\mathrm{H}} \top \\
\vdash_{\mathrm{H}} A \Rightarrow B \Rightarrow A \wedge B \\
\vdash_{\mathrm{H}} A \wedge B \Rightarrow A \\
\vdash_{\mathrm{H}} A \wedge B \Rightarrow B \\
\vdash_{\mathrm{H}} A \Rightarrow A \vee B \\
\vdash_{\mathrm{H}} B \Rightarrow A \vee B \\
\vdash_{\mathrm{H}} \bot \Rightarrow A \\
\vdash_{\mathrm{H}} (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow (A \vee B) \Rightarrow C
\end{array}$$

1.3 Classical logic

We replace $\bot \Rightarrow A$ by

$$\vdash^{c}_{\mathrm{H}} ((A \Rightarrow \bot) \Rightarrow \bot) \Rightarrow A$$

2 Natural deduction (N)

We view here contexts Γ as finite sets of formulas. The sequents have the form $\Gamma \vdash_{\mathrm{N}} A$. We write $\Gamma.A$ for $\Gamma \cup \{A\}$.

2.1 Implication

$$\frac{}{\Gamma.A \vdash_{\mathrm{N}} A}\ \mathrm{Hyp}$$

$$\frac{\Gamma.A \vdash_{\mathrm{N}} B}{\Gamma \vdash_{\mathrm{N}} A \Rightarrow B}\ {\Rightarrow_{\mathrm{I}}} \qquad \frac{\Gamma \vdash_{\mathrm{N}} A \Rightarrow B \qquad \Gamma \vdash_{\mathrm{N}} A}{\Gamma \vdash_{\mathrm{N}} B}\ {\Rightarrow_{\mathrm{E}}}$$

2.2 Propositional logic

$$\frac{}{\Gamma \vdash_{\mathrm{N}} \top}\ \top_{\mathrm{I}} \qquad \frac{\Gamma \vdash_{\mathrm{N}} A \qquad \Gamma \vdash_{\mathrm{N}} B}{\Gamma \vdash_{\mathrm{N}} A \wedge B}\ \wedge_{\mathrm{I}}$$

$$\frac{\Gamma \vdash_{\mathrm{N}} A \wedge B}{\Gamma \vdash_{\mathrm{N}} A}\ \wedge_{\mathrm{E1}} \qquad \frac{\Gamma \vdash_{\mathrm{N}} A \wedge B}{\Gamma \vdash_{\mathrm{N}} B}\ \wedge_{\mathrm{E2}}$$

$$\frac{\Gamma \vdash_{\mathrm{N}} A}{\Gamma \vdash_{\mathrm{N}} A \vee B}\ \vee_{\mathrm{I1}} \qquad \frac{\Gamma \vdash_{\mathrm{N}} B}{\Gamma \vdash_{\mathrm{N}} A \vee B}\ \vee_{\mathrm{I2}}$$

$$\frac{\Gamma \vdash_{\mathrm{N}} \bot}{\Gamma \vdash_{\mathrm{N}} A}\ \bot_{\mathrm{E}} \qquad \frac{\Gamma.A \vdash_{\mathrm{N}} C \qquad \Gamma.B \vdash_{\mathrm{N}} C \qquad \Gamma \vdash_{\mathrm{N}} A \vee B}{\Gamma \vdash_{\mathrm{N}} C}\ \vee_{\mathrm{E}}$$

2.3 Classical logic

We replace $\bot_{\mathrm{E}}$ by:

$$\frac{\Gamma.A \Rightarrow \bot \vdash_{\mathrm{N}}^{\mathrm{c}} \bot}{\Gamma \vdash_{\mathrm{N}}^{\mathrm{c}} A}\,\mathrm{RAA}$$

3 Sequent calculus (G)

For the intuitionistic sequent calculus we use the same judgements as for (N). The classical sequent calculus has judgements of the form $\Gamma \vdash_{\mathrm{G}} \Delta$, i..e with contexts on both sides.

3.1 Implication

$$\frac{}{\Gamma.A \vdash_{\mathrm{G}} A}\,\mathrm{Ax} \qquad \frac{\Gamma \vdash_{\mathrm{G}} A \qquad \Gamma.A \vdash_{\mathrm{G}} B}{\Gamma \vdash_{\mathrm{G}} B}\,\mathrm{Cut}$$

$$\frac{\Gamma.A \vdash_{\mathrm{G}} B}{\Gamma \vdash_{\mathrm{G}} A \Rightarrow B}\Rightarrow_{\mathrm{R}} \qquad \frac{\Gamma \vdash_{\mathrm{G}} A \qquad \Gamma.B \vdash_{\mathrm{G}} C}{\Gamma.A \Rightarrow B \vdash_{\mathrm{G}} C}\Rightarrow_{\mathrm{L}}$$

3.2 Propositional logic

$$\frac{}{\Gamma \vdash_{\mathrm{G}} \top}\,\top_{\mathrm{R}} \qquad \frac{\Gamma \vdash_{\mathrm{G}} A}{\Gamma.\top \vdash_{\mathrm{G}} A}\,\top_{\mathrm{L}} \qquad \frac{}{\Gamma.\bot \vdash_{\mathrm{G}} A}\,\bot_{\mathrm{L}}$$

$$\frac{\Gamma \vdash_{\mathrm{G}} A \qquad \Gamma \vdash_{\mathrm{G}} B}{\Gamma \vdash_{\mathrm{G}} A \wedge B}\,\wedge_{\mathrm{R}} \qquad \frac{\Gamma.A.B \vdash_{\mathrm{G}} C}{\Gamma.A \wedge B \vdash_{\mathrm{G}} C}\,\wedge_{\mathrm{L}}$$

$$\frac{\Gamma \vdash_{\mathrm{G}} A}{\Gamma \vdash_{\mathrm{G}} A \vee B}\,\vee_{\mathrm{R1}} \qquad \frac{\Gamma \vdash_{\mathrm{G}} B}{\Gamma \vdash_{\mathrm{G}} A \vee B}\,\vee_{\mathrm{R2}} \qquad \frac{\Gamma.A \vdash_{\mathrm{G}} C \qquad \Gamma.B \vdash_{\mathrm{G}} C}{\Gamma.A \vee B \vdash_{\mathrm{G}} C}\,\vee_{\mathrm{L}}$$

3.3 Classical logic

$$\frac{}{\Gamma.A \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A}\,\mathrm{Ax} \qquad \frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \qquad \Gamma.A \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} B}\,\mathrm{Cut}$$

$$\frac{\Gamma.A \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.B}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \Rightarrow B}\Rightarrow_{\mathrm{R}} \qquad \frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \qquad \Gamma.B \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}{\Gamma.A \Rightarrow B \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}\Rightarrow_{\mathrm{L}}$$

$$\frac{}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.\top}\,\top_{\mathrm{R}} \qquad \frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}{\Gamma.\top \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}\,\top_{\mathrm{L}} \qquad \frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.\bot}\,\bot_{\mathrm{R}} \qquad \frac{}{\Gamma.\bot \vdash_{\mathrm{G}}^{\mathrm{c}} A}\,\bot_{\mathrm{L}}$$

$$\frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \qquad \Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.B}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \wedge B}\,\wedge_{\mathrm{R}} \qquad \frac{\Gamma.A.B \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}{\Gamma.A \wedge B \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}\,\wedge_{\mathrm{L}}$$

$$\frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \vee B}\,\vee_{\mathrm{R1}} \qquad \frac{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.B}{\Gamma \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta.A \vee B}\,\vee_{\mathrm{R2}} \qquad \frac{\Gamma.A \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta \qquad \Gamma.B \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}{\Gamma.A \vee B \vdash_{\mathrm{G}}^{\mathrm{c}} \Delta}\,\vee_{\mathrm{L}}$$