# Logical relations and inductive/coinductive types

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**Abstract.** We investigate a $\lambda$ calculus with positive inductive and coinductive types , which we call $\lambda^{\mu,\nu}$, using logical relations. We show that parametric theories have the strong categorical properties, that the representable functors and natural transformations have the expected properties. Finally we apply the theory to show that terms of functorial type are almost canonical and that monotone inductive definitions can be reduced to positive in some cases.

## 1 Introduction

We investigate a $\lambda$ calculus with positive inductive and coinductive types, which we call $\lambda^{\mu,\nu}$, using logical relations. Here $\mu$ is used to construct the usual datatypes as initial algebras, where $\nu$-types are used for lazy types as terminal coalgebras.

A calculus related to $\lambda^{\mu,\nu}$ has for example been investigated by Geuvers [Geu92], but he does not consider nested $\mu$-types. Loader introduces the strictly positive fragment (and only $\mu$-types) in [Loa97] and shows that the theory of the PER model is maximal consistent. The restriction to the strict positive fragment gives rise to a predicative theory closely related to the theory $\mathbf{ID}_{<\omega}$ known from proof theory. The precise strength of the system of positive inductive definitions is unknown but can hardly be called predicative. Note that a proof by Buchholz [Buc81] that positive inductive definitions are conservative over strictly positive ones, does not apply to theories with parameters, i.e. to $\mu$-types with free type-variables. The contribution of $\nu$-types to the strength of the system seems peripheral but we shall not be concerned with this issue here. $\nu$-Types certainly have an important role for representing infinite structures in programming.

Logical relations have been well investigated in the context of System F to model Reynold's notion of *parametricity* [Rey83]. In his stimulating paper [Wad89] shows how to obtain *free theorems* and in [AP93] this is made precise by presenting a logic for parametricity. In particular it is shown show that parametricity entails that the 2nd order encodings of $\mu$ and $\nu$-types are strong, i.e. have the expected categorical properties. Several authors have presented categorical versions of parametricity [MR91,RR94,Has90]

The system we present here can be encoded in System F. However, we believe that it is of independent interest since inductive and coinductive definitions should be considered as more primitive than 2nd order quantification. In particular it is very straightforward to identify a predicative subsystem by restricting to strictly positive definitions. We present a notion of logical relations for inductive and coinductive types using least and greatest fixpoints on the metalevel, which means that the metatheory for the strict positive fragment itself is predicative and does not require 2nd order quantification.

One question which caused some confusion in the context of System F are the prerequisites for parametricity, i.e. the question whether Wadler's theorems are really free. I.e. the 2nd order encodings are not strong in the initial theory (not even if you restrict yourselves to closed terms), although it seems naively parametric. We say that a theory is parametric iff the identity extension property (not lemma) holds and show that all parametric theories are strong in the categorical sense.

We apply the theory to a question which came up in the work by Ralph Matthes [Mat98] on *monotone* inductive types, which is an apparent generalisation of positive. Matthes conjectured that good monotonicity witnesses can be already recognised by just testing the first functor law. It turns out that we can prove this conjecture rather easily using the theory of logical relations developed here. We also show that in the positive fragment the functions expressible with monotone inductive types are the same as in the standard theory. Everything dualises easily to the coinductive case. Another application of logical relations is the verification of the functor laws for the canonical map-terms. It relies on the fact that all terms with the type of a natural transformation are indeed natural which again can be easily verified using logical relations.

## 2 The calculus $\lambda^{\mu,\nu}$

Our starting point is simply typed $\lambda$-calculus over an infinite set of type variables $\mathrm{V}^{\mathrm{Ty}}$.

Let $S = \{+, -\}$ with the operation $- \in S \to S$ satisfying $-(+) = -, -(-) = +$. We simultaneously define the set of types Ty and the relation $\mathbf{OCC} \subseteq S \times \mathrm{V}^{\mathrm{Ty}} \times \mathrm{Ty}$ of positive and negative occurrence:

$$\frac{X \in \mathrm{V}^{\mathrm{Ty}}}{X \in \mathrm{Ty}} \qquad \frac{\sigma, \tau \in \mathrm{Ty}}{\sigma \to \tau, \sigma \times \tau, \sigma + \tau \in \mathrm{Ty}} \qquad \frac{X\mathbf{OCC}^{+}\sigma}{\mu X.\sigma, \nu X.\sigma \in \mathrm{Ty}}$$

$$\frac{}{X\mathbf{OCC}^{+}X} \qquad \frac{X \neq Y}{X\mathbf{OCC}^{s}Y}$$

$$\frac{X\mathbf{OCC}^{-s}\sigma \qquad X\mathbf{OCC}^{s}\tau}{X\mathbf{OCC}^{s}\sigma \to \tau} \qquad \frac{X\mathbf{OCC}^{s}\sigma \qquad X\mathbf{OCC}^{s}\tau}{X\mathbf{OCC}^{s}\sigma \times \tau \qquad X\mathbf{OCC}^{s}\sigma + \tau}$$

$$\frac{X\mathbf{OCC}^{s}\sigma \qquad Y\mathbf{OCC}^{+}\sigma}{X\mathbf{OCC}^{s}\mu Y.\sigma \qquad X\mathbf{OCC}^{s}\nu Y.\sigma}$$

We write $\boldsymbol{X}\mathbf{OCC}^{s}\sigma$ if $\boldsymbol{X} = X_1, \ldots, X_n$ s.t. for all $1 \leq i \leq n$: $X_i\mathbf{OCC}^{s}\sigma$.

The strictly positive system can be obtained by defining $-(-) = -$ instead, or alternatively by changing the rule for $\to$ to

$$\frac{X \notin \mathrm{FV}\sigma \qquad X\,\mathbf{OCC}^{+}\tau}{X\,\mathbf{OCC}^{+}\sigma \to \tau}$$

The definition of $\mathbf{OCC}^{-}$ can then be omitted.

Given a type $\sigma$ we say $\sigma \in \mathrm{Ty}(\boldsymbol{X})$ if all free type variables in $\sigma$ are included in the finite sequence $\boldsymbol{X}$. We write $\mathrm{Tm}(\sigma)$ for the set of closed terms of type $\sigma$.

Inspired by categorical notation we define

$$1_{\sigma} \equiv \lambda x : \sigma.x$$

and given $f : \rho \to \tau, g : \sigma \to \rho$

$$f \circ g \equiv \lambda x : \sigma.f(gx)$$

We may drop type annotations whenever they are uniquely determined from the context.

We write $\sigma(X)$ for a type in which the variable $X$ appears freely and then in the same context $\sigma(\tau)$ for the substitution $\sigma[X := \tau]$. This is extended to sequences, i.e. $\sigma(\boldsymbol{X})$ and $\sigma(\boldsymbol{\tau}) \equiv \sigma[\boldsymbol{X} := \boldsymbol{\tau}] \equiv \sigma[X_1 := \tau_1, \ldots, X_n := \tau_n]$ where we silently assume that the sequences involved have the right length.

Given $\sigma, \tau, \theta \in \mathrm{Ty}(\boldsymbol{X})$ and $\rho(X) \in \mathrm{Ty}(\boldsymbol{X}, X)$ with $X\,\mathbf{OCC}^{+}\rho(X)$ we introduce the following (families of) constants:

$$\begin{aligned}
\pi_1^{\sigma,\tau} &: (\sigma \times \tau) \to \sigma \\
\pi_2^{\sigma,\tau} &: (\sigma \times \tau) \to \tau \\
\mathrm{pair}^{\sigma,\tau} &: \sigma \to \tau \to (\sigma \times \tau) \\
\mathrm{i}_1^{\sigma,\tau} &: \sigma \to (\sigma + \tau) \\
\mathrm{i}_2^{\sigma,\tau} &: \tau \to (\sigma + \tau) \\
\mathrm{case}^{\sigma,\tau,\theta} &: (\sigma \to \theta) \to (\tau \to \theta) \to (\sigma + \tau) \to \theta \\
\mathrm{C}_{X.\rho(X)} &: \rho(\mu X.\rho) \to \mu X.\rho \\
\mathrm{It}^{\sigma}_{X.\rho} &: (\rho(\sigma) \to \sigma) \to (\mu X.\rho) \to \sigma \\
\mathrm{D}_{X.\rho(X)} &: \nu X.\rho \to \rho(\nu X.\rho) \\
\mathrm{Co}^{\sigma}_{X.\rho(X)} &: (\sigma \to \rho(\sigma)) \to \sigma \to (\nu X.\rho)
\end{aligned}$$

We define the empty type $\emptyset \equiv \mu X.X$ and the unit type $\mathbf{1} \equiv \nu X.X$ and $* = \mathrm{Co}_{X.X}\ 1_{\emptyset \to \emptyset} 1_{\emptyset} : \mathbf{1}$ as the canonical inhabitant of $\mathbf{1}$. We write $\mathbf{2} = \mathbf{1} + \mathbf{1}$ for the type of booleans and $\mathrm{true} = \mathrm{i}_1\ *, \mathrm{false} = \mathrm{i}_2\ *$.

We are now going to define internal functors (or map terms) for positive and negative type abstractions. Given $\boldsymbol{X}\mathbf{OCC}^{+}\rho(\boldsymbol{X};\boldsymbol{Y}), \boldsymbol{Y}\mathbf{OCC}^{-}\rho(\boldsymbol{X},\boldsymbol{Y})$, for $1 \le i \le |\boldsymbol{X}|$: $f_i : \sigma_i \to \sigma'_i$ and for $1 \le j \le |\boldsymbol{Y}|$: $g_j : \tau_j \to \tau'_j$ we define

$$\rho(\boldsymbol{f};\boldsymbol{g}) : \rho(\boldsymbol{\sigma};\boldsymbol{\tau'}) \to \rho(\boldsymbol{\sigma'};\boldsymbol{\tau})$$

by induction over the structure of $\rho$:

$$\begin{aligned}
X_i(\boldsymbol{f};\boldsymbol{g}) &= f_i \\
Z(\boldsymbol{f};\boldsymbol{g}) &= 1_Z \\
(\rho \to \rho')(\boldsymbol{f};\boldsymbol{g}) &= \lambda g.\rho'(\boldsymbol{f};\boldsymbol{g}) \circ g \circ \rho(\boldsymbol{g};\boldsymbol{f}) \\
(\rho \times \rho')(\boldsymbol{f};\boldsymbol{g}) &= \lambda p.\mathrm{pair}(\rho(\boldsymbol{f};\boldsymbol{g})(\pi_1\ p))(\rho'(\boldsymbol{f};\boldsymbol{g})(\pi_2\ p)) \\
(\rho + \rho')(\boldsymbol{f};\boldsymbol{g}) &= \lambda s.\mathrm{case}\ (\mathrm{i}_1 \circ \rho(\boldsymbol{f};\boldsymbol{g}))(\mathrm{i}_2 \circ \rho'(\boldsymbol{f};\boldsymbol{g})) \\
(\mu X.\rho)(\boldsymbol{f};\boldsymbol{g}) &= \mathrm{It}_{X.\rho}\ (\mathrm{C} \circ \rho(1,\boldsymbol{f};\boldsymbol{g})) \\
(\nu X.\rho)(\boldsymbol{f};\boldsymbol{g}) &= \mathrm{Co}_{X.\rho}(\rho(1,\boldsymbol{f};\boldsymbol{g}) \circ \mathrm{D})
\end{aligned}$$

A theory $\sim$ is a family of equivalence relations $\sim_\sigma \subseteq \mathrm{Tm}(\sigma) \times \mathrm{Tm}(\sigma)$ which is a congruence for application and closed under $\beta$[1]: $(\lambda x : \sigma.t)u \sim t[x := u]$. Additionally we assume the following equivalences:

$$\begin{aligned}
\pi_1^{\sigma,\tau}(\mathrm{pair}^{\sigma,\tau}\ t\ t') &\sim t \\
\pi_2^{\sigma,\tau}(\mathrm{pair}^{\sigma,\tau}\ t\ t') &\sim t' \\
(\mathrm{case}\ t\ u) \circ \mathrm{i}_1 &\sim t \\
(\mathrm{case}\ t\ u) \circ \mathrm{i}_2 &\sim u \\
(\mathrm{It}_{X.\rho} t) \circ \mathrm{C}_{X.\rho} &\sim t \circ \rho(\mathrm{It}_{X.\rho} t) \\
\mathrm{D}_{X.\rho} \circ (\mathrm{Co}_{X.\rho} t) &\sim \rho(\mathrm{Co}_{X.\rho} t) \circ t
\end{aligned}$$

In general we assume that a theory is consistent, i.e. not all well typed equations on closed terms hold. The initial or least theory is denoted by $\sim^\beta$.

For $\sim^\beta$ a strongly normalising and confluent term rewriting system can be found. These properties can be verified by developing the notion of logical predicates corresponding to the logical relations presented below, see [Mat98]. The main corollary of this is:

**Proposition 1.** *$\sim^\beta$ is decidable for typed terms.*

### 2.1 Examples

We can embed System T by defining $\mathrm{Nat} = \mu X.\mathbf{1}+X$ with $0 = \mathrm{C}_{X.\mathbf{1}+X}(\mathrm{i}_1 *)$, $\mathrm{s} = \mathrm{C}_{X.\mathbf{1}+X} \circ \mathrm{i}_2$. From It we can derive the standard iterator for Nat. Also a recursor $\mathrm{R} : \sigma \to (\mathrm{Nat} \times \sigma \to \sigma) \to \mathrm{Nat} \to \sigma$ is definable using the iterator. R does not satisfy the usual equations for a recursor. However, it can be shown that this encoding is correct in extensional theories, to be defined below.

Using $\nu$-types we can define the *conatural* numbers $\mathrm{Nat}' = \nu X.\mathbf{1} + X$, which intuitively are the natural numbers extended by $\omega = \mathrm{Co}^{X.\mathbf{1}+X,\mathbf{1}}\mathrm{i}_2 *$.

Other examples are: lists over $\sigma$:

$$\mathrm{L}^\sigma = \mu X.\mathbf{1} + \sigma \times X,$$

[1] Note that we do not include $\xi$ or $\eta$, i.e. our starting point is an extension of typed combinatory logic. Later we will introduce extensional theories which are closed under $\xi$ and $\eta$ for all type formers.

infinite streams over $\sigma$:

$$\mathrm{S}^\sigma = \nu X.\sigma \times X,$$

finitely branching trees:

$$\mathrm{F} = \mu X.\mathrm{L}^X.$$

The last definition is interesting because it uses a $\mu$-type with a free type variable. We call this an inductive definition with parameters[2]. It seems that the counterpart of inductive definition with parameters is (intentionally ?) excluded from the standard definitions of ID, e.g.[Buc81].

We define the type of trees branching over $\sigma$:

$$\mathrm{T}^\sigma = \mu X.1 + \sigma \to X,$$

e.g. $\mathrm{T}^\mathbf{2}$ are binary trees. Using T we can define a hierarchy of tree types:

$$\mathrm{T}_0 = \emptyset$$
$$\mathrm{T}_{n+1} = \mathrm{T}^{\mathrm{T}_n}$$

The hierarchy of trees exhausts the proof-theoretic strength of $\mathrm{ID}_{<\omega}$, however in the presence of non-strict positive definitions with parameters, we can define

$$\mathrm{T}^\infty = \mu X.\mathrm{T}^{\mathrm{T}^X}$$

This type may raise some doubts in whether this system is predicative (i.e. proof theoretically conservative over the strict positive system).

## 3 Logical Relations

We consider here $n$-ary relations on closed terms which have to be closed under $\sim^\beta$. Given $\boldsymbol{\sigma} = \sigma_1, \ldots, \sigma_n$ a sequence of closed types, we write $R \in \mathrm{Rel}(\boldsymbol{\sigma})$ for $R \subseteq \mathrm{Tm}(\sigma_1) \times \ldots \times \mathrm{Tm}(\sigma_n)$. We define an operator $\tau^\#(-)$ which to any sequence $\boldsymbol{R} = R_1, \ldots, R_m$ with $R_i \in \mathrm{Rel}(\boldsymbol{\sigma_i})$ assigns

$$\tau^\#(\boldsymbol{R}) \in \mathrm{Rel}(\tau(\boldsymbol{\sigma_1}), \ldots, \tau(\boldsymbol{\sigma_m}))$$

At the same time we have to verify that if $X_i\mathbf{OCC}^+(\mathbf{OCC}^-)\rho$ then $\rho^\#(-)$ is monotone(antitone) in its $i$th argument.

We extend application to tuples:

$$\boldsymbol{t}\ \boldsymbol{u} = t_1\ u_1, \ldots, t_n\ u_n$$

and

$$c\ \boldsymbol{u} = c\ u_1, \ldots, c\ u_n$$

where $c$ is a constant.

[2] [Mat98] calls this *interleaving inductive definitions.*

**Projection**

$$X_i^{\#}(\boldsymbol{R})\ \boldsymbol{t} :\Longleftrightarrow\ R_i\ \boldsymbol{t}$$

$\to$

$$(\sigma \to \tau)^{\#}(\boldsymbol{R})\ \boldsymbol{t} :\Longleftrightarrow\ \forall \boldsymbol{u}.\sigma^{\#}(\boldsymbol{R})\boldsymbol{u} \to \tau^{\#}(\boldsymbol{R})(\boldsymbol{t}\ \boldsymbol{u})$$

$\times$

$$(\sigma \times \tau)^{\#}(\boldsymbol{R})\ \boldsymbol{t} :\Longleftrightarrow\ \sigma^{\#}(\boldsymbol{R})(\pi_1 \boldsymbol{t}) \wedge \tau^{\#}(\boldsymbol{R})(\pi_2 \boldsymbol{t})$$

$+$ $(\sigma_1 + \sigma_2)^{\#}(\boldsymbol{R})$ is the least relation generated by

$$\frac{\sigma_k^{\#}(\boldsymbol{R})\boldsymbol{u}}{(\sigma_1 + \sigma_2)^{\#}(\boldsymbol{R})(\mathrm{i}_k\ \boldsymbol{u})} +^{\#}$$

$\mu$ Given $\mu X.\rho$ we know that $X\mathbf{OCC}^{+}\rho$ and hence $\rho^{\#}(\boldsymbol{R}, -)$ is monotone. We define $(\mu X.\rho)^{\#}$ as the least relation closed under

$$\frac{\rho^{\#}(\boldsymbol{R}, (\mu X.\rho)^{\#})\ \boldsymbol{t}}{(\mu X.\rho)^{\#}(\mathrm{C}_{X.\rho}\ \boldsymbol{t})} \mu^{\#}$$

$\nu$ Given $\nu X.\rho$ we know that $X\mathbf{OCC}^{+}\rho$ and hence $\rho^{\#}(\boldsymbol{R}, -)$ is monotone. We define $(\nu X.\rho)^{\#}$ as the greatest relation closed under

$$\frac{(\nu X.\rho)^{\#}\ \boldsymbol{t}}{\rho^{\#}(\boldsymbol{R}, (\nu X.\rho)^{\#})\ (\mathrm{D}_{X.\rho}\ \boldsymbol{t})} \nu^{\#}$$

**Proposition 2 (Fundamental property of logical relations).** *Given $\sigma(\boldsymbol{X}) \in \mathrm{Ty}(\boldsymbol{X})$ , $t(\boldsymbol{X}) : \sigma(\boldsymbol{X})$ and $\boldsymbol{R} : \boldsymbol{\tau_1} \ldots \boldsymbol{\tau_m}$ we have that*

$$\sigma^{\#}(\boldsymbol{R})\ (t(\boldsymbol{\tau_1}), \ldots, t(\boldsymbol{\tau_m}))$$

*Proof.* To simplify notation we consider only the binary case here (and use infix notation for the relation). The proof proceeds by induction on (the derivation of) $t(\boldsymbol{X}) : \sigma(\boldsymbol{X})$. Since the proof for simply typed $\lambda$ terms is standard (we only have to check combinatory logic), we concentrate on the new constants:

$\times$,$+$ We just do $+$: The case for $\mathrm{i}_k$ follows directly from $+^{\#}$. Assume $f_k(\sigma_k \to \theta)^{\#}(\boldsymbol{R})f_k'$ and $t(\sigma_1 + \sigma_2)^{\#}(\boldsymbol{R})t'$. By the definition of $(\sigma_1 + \sigma_2)^{\#}(\boldsymbol{R})$ we have that $t \sim \mathrm{i}_k\ u$, $t' \sim \mathrm{i}_k\ u'$ and $u\sigma_k^{\#}(\boldsymbol{R})u'$. Using case $f_1 f_2(\mathrm{i}_k u) \sim f_k\ u$ (and analogously for $f_k', u'$) we get case $f_1 f_2(\mathrm{i}_k u)\theta^{\#}(\boldsymbol{R})$case $f_1' f_2'(\mathrm{i}_k u')$ and hence the property for case.

$\mu$,$\nu$ The case for C follows directly from $\mu^{\#}$. Assume $f(\rho(\sigma) \to \sigma)^{\#}(\boldsymbol{R})f'$ define $R \in \mathrm{Rel}(\mu X.\rho(X,\tau), \mu X.\rho(X,\tau'))$ by $tRt' :\Longleftrightarrow$ It $f\ t\sigma^{\#}(\boldsymbol{R})$It $f'\ t'$. Using the $\beta$-equality for $\mu$ it is easy to see that $R$ is closed under $\mu^{\#}$. Hence by minimality we have that $t(\mu X.\rho)^{\#}(\boldsymbol{R})t'$ implies $tRt'$ which directly entails the correctness for It. The case for $\nu$ is an easy dualisation.

□

## 4 Extensional Theories

Using logical relations we show properties of extensional theories, that is theories in which semantically equivalent terms are identified. We show that parametric theories have the universal categorical properties.

**Definition 1 (Strong Theories).** *We call a theory* strong *iff it satisfies the usual universal properties of function spaces, products, coproducts, initial algebras and terminal coalgebras. Equivalently the theory $\sim$ has to be closed under the usual $\xi_{\rightarrow}$ and $\rightarrow_{\eta}$ rules and:*

$$\frac{}{\text{pair } (\pi_1\ t)\ (\pi_2 t) \sim t}\ \eta_\times$$

*Note that there is no $\xi_\times$ because the appropriate equality follows from the fact that application is a congruence.*

$$\frac{}{(\text{case } \mathrm{i}_1\ \mathrm{i}_2) \sim 1}\ \eta_+ \qquad \frac{}{u \circ (\text{case } t_1\ t_2) \sim \text{case } (u \circ t_1)\ (u \circ t_2)}\ \xi_+$$

$$\frac{}{\mathrm{It}^{X.\rho}\mathrm{C}_{X.\rho} \sim 1}\ \eta_\mu \qquad \frac{\begin{array}{c} f : \rho(\tau) \to \tau \\ g : \rho(\sigma) \to \sigma \\ h : \tau \to \sigma \\ h \circ f \sim g \circ \rho(h) \end{array}}{h \circ (\mathrm{It}\ f) \sim \mathrm{It}\ g}\ \xi_\mu$$

$$\frac{}{\mathrm{Co}_{X.\rho}\mathrm{D}_{X.\rho} \sim 1}\ \eta_\nu \qquad \frac{\begin{array}{c} f : \tau \to \rho(\tau) \\ g : \sigma \to \rho(\sigma) \\ h : \sigma \to \tau \\ f \circ h \sim \rho(h) \circ g \end{array}}{(\mathrm{Co}\ f) \circ h \sim \mathrm{Co}\ g}\ \xi_\nu$$

We here define parametric theories using $n$-ary relations.

**Definition 2 (Parametric theories).** *We define the $n$-diagonal as*

$$\Delta^n_\sigma = \{(t_1, \ldots, t_n) \mid \forall 1 \leq i, j \leq n. t_i \sim t_j\} \in \mathrm{Rel}(\sigma^n)$$

.

*We call a theory parametric iff it satisfies the identity extension property, that is for any $\tau(\boldsymbol{X}) \in \mathrm{Ty}(\boldsymbol{X})$ we have:*

$$\tau^{\#}(\Delta^n_{\sigma_1}, \ldots, \Delta^n_{\sigma_m}) = \Delta^n_{\tau(\boldsymbol{\sigma})}$$

*In the case $n = 2$ this is just*

$$t\tau^{\#}(\sim_{\boldsymbol{\sigma}})t' \iff t \sim_{\sigma(\boldsymbol{\tau})} t'$$

**Definition 3 (Observational equivalence).** *We define the observational equivalence by*

$$t \sim^{\text{obs}}_\sigma u \iff \forall p : \sigma \to \mathbf{2}. p\ t \sim^\beta p\ u$$

The following proposition is folklore and goes back to Statman for simply type $\lambda$ calculus and Moggi [Mog] in the case of System F. In fact it can be extended to all theories which have a type of booleans with a polymorphic if.

**Proposition 3.** *The observational congruence is the greatest consistent theory, that is if $t \sim u$ in any (consistent) theory then $t \sim^{\mathrm{obs}} u$*

*Proof.* To see that obs is actually a theory note that $t \sim^{\mathrm{obs}} t' \to tu \sim^{\mathrm{obs}} t'u$ and symmetrically. It is easy to see that obs is closed by the defining equations for $\sim^{\beta}$. To see that $\sim^{\mathrm{obs}}$ is consistent use $\mathbf{1_2}$ to see that true $\not\sim^{\mathrm{obs}}$ false.

Given any consistent theory with $t \sim_{\sigma} t'$. Assume that there is a $p : \sigma \to \mathbf{2}$ s.t. $p\ t \not\sim^{\beta} p\ t'$. W.l.o.g assume $p\ t \sim^{\beta}$ true and $p\ t \sim^{\beta}$ false but then true $\sim p\ t \sim p\ t' \sim$ false and hence $\sim$ is not consistent. $\square$

In the following we use $\sim$ for any parametric equality.

## 5 Main results

Given $t : \sigma \to \tau$ we define the graph of $t$: $t^{\#} : \sigma \iff \tau$ by $u\, t^{\#}\, u' :\iff u' \sim tu$. We show at once the graph theorem, naturality of certain functions and the functor law, because all these results depend upon each other — the situation is very different from System F [AP93].

**Proposition 4 (Map properties).** *Assume*

$$\boldsymbol{X}\mathbf{OCC}^{+}\rho(\boldsymbol{X};\boldsymbol{Y},\boldsymbol{Z}), \boldsymbol{Y}\mathbf{OCC}^{-}\rho(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}).$$

1. $\rho^{\#}(f^{\#};{g^{\#}}^{-1};\sim) \iff \rho(f;g;\sim)$
2. *For $t(X) : \rho(X) \to \rho'(X), f : \sigma \to \tau$ it holds:*

$$t(\tau) \circ \rho(f) \sim \rho'(f) \circ t(\sigma)$$

3. *The $\eta$-rules are valid.*
4. *the $\mu$-rules are valid.*
5. $\rho(1_{\sigma}) \sim 1_{\rho(\sigma)}$
6. $\rho(f \circ g; h \circ k) \sim \rho(f;k) \circ \rho(g;h)$

*Proof.* We show 1-6 by parallel induction over the size of $\rho, \rho'$. We only consider the case of single argument covariant functors here.

1. The cases for the basic type constructors $\to, \times, +$ are straightforward. We discuss the case for $\mu$ in some detail: Let $\rho(X) = \mu Y.\theta(Y, X)$. We have to show:
   (a) $x\rho^{\#}(f^{\#})y \to x\rho(f)^{\#}y$
   We show that

$$x\theta^{\#}(\rho(f)^{\#}, f^{\#})y \to (\mathrm{C}\ x)\rho(f)^{\#}(\mathrm{C}\ y) \tag{1}$$

From this the proposition above follows from the fact that $\rho^{\#}(f^{\#})$ is the least relation with this property. To show 1 assume $x\theta^{\#}(\rho(f)^{\#}, f^{\#})y$ which by ind.hyp. implies $y \sim \theta(\rho(f), f)\ x$. We want to show

$$(\mathrm{C}\ x)\rho(f)^{\#}(\mathrm{C}\ y)$$

which is equivalent to

$$\begin{aligned}\mathrm{C}\ y &\sim \rho(f)(\mathrm{C}\ x)\\ &\sim \mathrm{C} \circ \theta(1, f) \circ \theta(\rho(f), 1)\\ &\sim \mathrm{C} \circ \theta(\rho(f), f)\end{aligned}$$

which follows from the assumption.

(b) $x\rho(f)^{\#}y \to x\rho^{\#}(f^{\#})y$
We show that $P(x) = x\rho^{\#}(f^{\#})\rho(f)\ x$ is closed under the condition for (unary) logical predicates, that is $\rho^{\#}(P)(cx) \to P\ x$ using the closure condition for $\rho^{\#}(f^{\#})$.

2. Parametricity implies

$$x\rho^{\#}(f^{\#})y \to \alpha(\sigma)\ x\rho'^{\#}(f^{\#})\alpha(\sigma)\ x$$

Using 1. we obtain:

$$\rho'(f) \circ \alpha(\sigma) \sim \alpha(\tau) \circ \rho(f)$$

3. We just show $\eta_\mu$: We show that $R = (\mathrm{It}^{X.\rho}\ \mathrm{C})^{\#}$ is closed under $\mu^{\#}$. Hence assume $x\rho^{\#}(R)y$. We have to show $\mathrm{C}\ x\ R\ \mathrm{C}\ y$ that is $(\mathrm{It}\ \mathrm{C})(\mathrm{C}x) \sim (\mathrm{C}y)$. We note that $(\mathrm{It}\ \mathrm{C})(\mathrm{C}x) \sim \mathrm{C}(\rho(\mathrm{It}\ \mathrm{C})x)$ and hence we can reduce the problem to the hypothesis. Using the minimality of $\mu^{\#}$ we conclude $x \sim y \to \mathrm{It}\ \mathrm{C}\ x \sim y$ and hence $\mathrm{It}\ \mathrm{C}\ \sim 1$.
4. Similar to the previous case.
5. Follows directly from the $\eta$-rules.
6. For $\to$,$+$ and $\times$ the composition laws follow directly from the $\xi$-rules. However for $\mu$ and $\nu$ the situation is more complicated. Let's consider $\rho(X) = \mu Y.\theta(X, Y), g : \sigma_1 \to \sigma_2, f : \sigma_2 \to \sigma_3$,:

$$\begin{aligned}\rho(f \circ g) &\sim \mathrm{It}(\mathrm{C} \circ \theta(\rho(\sigma_3), f \circ g)\\ \rho(f) \circ \rho(g) &\sim (\mathrm{It}(\mathrm{C} \circ \theta(\rho(\sigma_3), f))) \circ (\mathrm{It}(\mathrm{C} \circ \theta(\rho(\sigma_2), g)))\end{aligned}$$

Let $h = \mathrm{It}(\mathrm{C} \circ \theta(\rho(\sigma_3), f))$ To show that both sides are equal we apply $\xi_\mu$ (ind.hyp. 4.) which reduces the problem to showing the equivalence of

$$\mathrm{C} \circ \theta(\rho(\sigma_3), f \circ g) \circ \theta(h, \sigma_1)$$

and

$$h \circ \mathrm{C} \circ \theta(\rho(\sigma_2), g).$$

Using $\sim_\beta$ on the second term we obtain

$$\mathrm{C} \circ \theta(\rho(\sigma_3), f) \circ \theta(h, \sigma_3) \circ \theta(\rho(\sigma_2), g)$$

Now using naturality (ind.hyp. 2.) we can show that the term is equivalent to

$$\mathrm{C} \circ \theta(\rho(\sigma_3), f) \circ \theta(\rho(\sigma_2), g) \circ \theta(h, \sigma_1)$$

and by applying the ind.hyp. 6. we show the equivalence to the first term. □

Note that [Geu92] does not prove the functor laws for nested $\mu$-types, which is the difficult case. The use of logical relations and naturality may be avoided by an exhaustive case analysis[3] but the present proof seems much more elegant.

**Corollary 1.** *Parametric theories are strong.*

## 6 Monotone inductive types

We now turn our attention to general properties of terms having the type of map-terms. It turns out that those are almost canonical.

**Proposition 5.** *Given $X\mathbf{OCC}^+\rho(X)$ and*

$$m(X,Y) : (X \to Y) \to \rho(X) \to \rho(Y)$$

*we set*

$$\alpha(X) = m(X,X)1_X : \rho(X) \to \rho(X)$$

*We can show that*

$$\begin{aligned} m(\sigma,\tau)f &= \rho(f) \circ \alpha(\sigma) \\ &= \alpha(\tau) \circ \rho(f) \end{aligned}$$

*Proof.* The fundamental theorem entails that for any $S : \sigma \iff \sigma'$ and $R : \tau \iff \tau'$ and for $h : \sigma \to \tau, h' : \sigma' \to \tau'$ such that $tSt'$ implies $(f\ t)\ R\ (f'\ t')$ and $a\rho^{\#}(S)a'$ it is the case that

$$(m(\sigma,\tau)\ f\ a)\rho^{\#}(R)\ (m(\sigma',\tau')\ f'\ a')$$

We now set $f' = 1, S = \sim, R = f^{\#}$ and by using the map theorem we obtain the first line. The second line follows by a symmetric argument. □

We can immediately conclude the following corollary which was conjectured by Ralph Matthes in the context of monotone inductive definitions:

**Corollary 2.** *Given a map term for $X\mathbf{OCC}^+\rho(X)$*

$$m(X,Y) : (X \to Y) \to \rho(X) \to \rho(Y)$$

[3] Suggested by Ralph Matthes.

*which satisfies the first functor law*

$$m(X, X)1_X = 1_{\rho(X)}$$

*Then $m$ is already equal to the proper map:*

$$m(X, Y) \sim \lambda f.\rho(f)$$

We introduce the concept of monotone inductive definitions as a generalisation of positive inductive definitions, i.e. we drop the positivity requirement but parametrise the $\mu$-types with a *monotonicity* witness:

Abbreviate the type of monotonicity witnesses for $\rho(X)$ as $\phi_\rho(X, Y) = (X \to Y) \to (\rho(X) \to \rho(Y))$

$$\frac{m(X, Y) : \phi_\rho(X, Y)}{\mu_m X.\rho(X) \in \mathrm{Ty}}$$

$$\begin{aligned} \mathrm{C}_{m,X.\rho} &: \rho(\mu_m X.\rho) \to \mu_m X.\rho \\ \mathrm{It}^{\sigma}_{m,X.\rho} &: (\rho(\sigma) \to \sigma) \to (\mu_m X.\rho) \to \sigma \\ (\mathrm{It}^{\sigma}_{m,X.\rho} t) \circ \mathrm{C}_{m,X.\rho} &\sim t \circ (m(\mu_m X.\rho(X), \sigma(\mathrm{It}^{X.\rho,\sigma} t)) \end{aligned}$$

This presentation is appealing because we do not have to define the functors anymore. Moreover we can show:

**Corollary 3.** *Given $X\mathbf{OCC}^+\rho(X)$ and any $m(X, Y) : \phi_\rho(X, Y)$, $f : \rho(\sigma) \to \sigma$ then we have:*

$$\mathrm{It}^{X.\rho,\sigma}_m \sim \mathrm{It}^{X.\rho,\sigma}((m(\sigma, \sigma)1_\sigma) \circ f)$$

This implies that for positive $\rho$ everything we can define for monotone types is already definable in the standard theory. The analogous facts hold about monotone $\nu$-types.

Note that we do not say anything about the case when $X$ does not appear positively in $\rho(X)$. Indeed, there are interesting examples of monotone but not positive type abstractions like the following monotone type (discovered by Ulrich Berger):

$$\mu_m X.(((X \to \sigma) \to X) \to X) \to X$$

However, it seems that no such $m$ satisfies the first functor law, and we conjecture that they do not increase the computational strength of the system.

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