A syntax for cubical type theory (draft)

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In this paper we provide a syntax for the cubical set model of type theory [3].

We start by defining a heterogeneous equality as a logical relation in an extended context (section 1.1). This can be seen as a different presentation of parametricity for dependent types [1]. We investigate the higher dimensional structure induced by the logical relation in section 1.2. The relation defined so far is a very weak equality which is not even reflexive, so as a first step, we add reflexivity (section 1.3). This can be seen as a different presentation of the internalisation of parametricity [2]. To add symmetry and transitivity of equality, and in general, to prove the J eliminator, we modify our theory in one more step by adding Kan fillers to the universe (1.4). This makes our construction an internalisation of the presheaf model given by cubical sets (section 2). Finally we prove some metatheoretic properties of the system 3.

The parts which are missing:

- We haven't yet defined uncoe and uncoh for each type former, see end of section 1.4.1.
- It is not clear how to swap the universe with the new definition of U^*, see section 1.4.5.
- All the numbered equations should be proven in section 3.
- Prove decidability of definitional equality, canonicity.

1 Syntax

We start with the usual rules of type theory with explicit substitutions without the inductive identity type and $\mathsf{U} : \mathsf{U}$ for simplicity (like in [3]) and add additional rules which define the identity type as a logical relation. When we say things like "defining by induction on the term structure" this is only for intuition and does not mean the definition of a meta-operation on the syntax: we are just adding new rules to the theory.

Notation for substitutions: if $\rho : \Delta \to \Gamma$ (list of terms of types Γ having free variables in Δ), $\Gamma \vdash a : A$, then $\Delta \vdash a[\rho] : A[\rho]$.

If $\vdash \Gamma$ and Δ is a context depending on Γ (denoted $\vdash \Gamma.\Delta$), we sometimes write $\Gamma \vdash \Delta$; and just as $\Gamma \vdash a : A$ is an abbreviation for $(\mathsf{id}_\Gamma, a) : \Gamma \to \Gamma.A$, we write $\Gamma \vdash \rho : \Delta$ for $(\mathsf{id}_\Gamma, \rho) : \Gamma \to \Gamma.\Delta$.

1.1 Heterogeneous equality

In this section we define heterogeneous equality as a logical relation. This can be seen as a different presentation of [1].

Logical relations provide a convenient framework to talk about equalities depending on other equalities. In type theory with the inductive identity type, if $e : A = B$, $a : A$, $b : B$, then we can talk about equality of a and b as an element of the type $\mathsf{coe}\, e\, a =_B b$. But this does not generalise nicely to higher dimensions: eg. if we also know that c and d are equal heterogeneously, then how do we talk about equalities of these heterogeneous equalities? More precisely, we are in the following situation, the types are on the right hand side, the elements on the left:

$$\begin{tikzcd} c \arrow[r, "q"] & d \\ a \arrow[u, "r"] \arrow[r, "p"] & b \arrow[u, "s"'] \end{tikzcd} \quad : \quad \begin{tikzcd} C \arrow[r, "f"] & D \\ A \arrow[u, "g"] \arrow[r, "e"] \arrow[ur, phantom, "m"] & B \arrow[u, "h"'] \end{tikzcd}$$

So, eg. $q : \mathsf{coe}\, f\, c =_D d$, $r : \mathsf{coe}\, g\, a =_C c$ etc. Now we can state the equality of p and q as an element of the following type:

$$\mathsf{transport}\, (\lambda z.c = z)\, s\, (\mathsf{transport}\, (\lambda z.z = b))\, r\, p) =_{c=d} q$$

In the notation introduced below, $\mathsf{coe}\, e\, a =_B b$ becomes $a \sim_e b$. To state the equality of p and q, we need a type of squares $m : e \sim_{(=_{\mathsf{U}})^*\, A\, C\, g\, B\, D\, h} f$, and now we can denote the desired type as $p \sim_{(\sim_m)^*\, a\, c\, r\, b\, d\, s}\, q$.

We define an operation $\text{–}^=$ on contexts and terms mutually to justify the following rule:

$$\frac{\Gamma \vdash u : A}{\Gamma^= \vdash u^= : A^=} \tag{1}$$

Note that $A^=$ will be a context, hence $u^=$ is a sequence of terms, part of the substitution $(\mathsf{id}, u^=) : \Gamma^= \to \Gamma^=.A^=$.

1.1.1 $\text{–}^=$ on contexts

$\text{–}^=$ triples a context: for a type A in the context, it provides two copies of that type (substituted as required) and a witness that the first two elements are related in the relation $- \sim_{A^*} -$ which will be defined later.

$$\begin{aligned} \emptyset^= \quad & \equiv \emptyset \\ (\Gamma.x : A)^= & \equiv \Gamma^=.(x : A)^= \equiv \Gamma^=.x_0 : A[\mathsf{0}_\Gamma].x_1 : A[\mathsf{1}_\Gamma].x_2 : x_0 \sim_{A^*} x_1 \end{aligned}$$

$\mathsf{0}$ and $\mathsf{1}$ are substitutions which project out the corresponding elements from the tripled context: for $i \in \{\mathsf{0}, \mathsf{1}\}$, $a[i]$ is the term a where each free variable x in a is replaced by x_i.

$$\begin{aligned} i_\emptyset \quad & \equiv () \qquad : \emptyset \to \emptyset \\ i_{\Gamma.A} & \equiv (i_\Gamma, x_i) : \Gamma^=.x_0 : A[\mathsf{0}].x_1 : A[\mathsf{1}].x_2 : x_0 \sim_{A^*} x_1 \to \Gamma.A \end{aligned}$$

With logical relation lenses, $\Gamma^=$ can be seen as the context which contains two copies of Γ and also the proof that they are related.

1.1.2 $-^{=}$ on terms

$-^{=}$ is defined on terms by substituting the term u by $\mathsf{0}$ and $\mathsf{1}$, and then providing a witness u^* that these are related:

$$\frac{\Gamma \vdash u : A}{\Gamma^{=} \vdash u^{=} \equiv (u[\mathsf{0}_\Gamma], u[\mathsf{1}_\Gamma], u^*) : \underbrace{A^{=}}_{\equiv x_0 : A[\mathsf{0}_\Gamma].x_1 : A[\mathsf{1}_\Gamma].x_2 : x_0 \sim_{A^*} x_1}} \tag{2}$$

Now we only need to define the last element of the tripled context to justify the following rule:

$$\frac{\Gamma \vdash u : A}{\Gamma^{=} \vdash u^* : u[\mathsf{0}_\Gamma] \sim_{A^*} u[\mathsf{1}_\Gamma]} \tag{3}$$

$\sim_{A^*}$ can be viewed as a heterogeneous equality relation between types $A[\mathsf{0}]$ and $A[\mathsf{1}]$. With this view, 3 expresses that every term is a congruence. With logical relation lenses, rule 3 expresses parametricity: the interpretations of the term u in different contexts ($u[\mathsf{0}]$ and $u[\mathsf{1}]$) are related in the relation generated by it's type ($\sim_{A^*}$).

We will define u^* by induction on the term structure of u: in section 1.1.3 we define U^*, in section 1.1.4 we define A^* for other types $A : \mathsf{U}$, and in section 1.1.5 we define t^* for all non-inhabitable terms t.

For now, we define $\sim$ as the identity function, so that $\sim_{A^*} \equiv A^*$ and we have

$$\frac{\Gamma^{=} \vdash u : A[\mathsf{0}] \quad \Gamma^{=} \vdash v : A[\mathsf{0}]}{\Gamma^{=} \vdash u \sim_{A^*} v \equiv A^*\, u\, v : \mathsf{U}}$$

In section 1.4 we will add more components into A^*, and $\sim$ will become a projection function.

1.1.3 $-^*$ on the universe

We define U^* in a way that makes it obey rule 3. On the left side we write the corresponding rule in the original type theory.

$$\frac{}{\Gamma \vdash \mathsf{U} : \mathsf{U}} \qquad \frac{}{\Gamma \vdash \mathsf{U}^* \equiv \lambda A\, B.A \to B \to \mathsf{U}}$$

So we have $\mathsf{U}^* : \mathsf{U}[\mathsf{0}] \sim_{\mathsf{U}^*} \mathsf{U}[\mathsf{1}] \equiv \mathsf{U}^*\, \mathsf{U}\, \mathsf{U} \equiv \mathsf{U} \to \mathsf{U} \to \mathsf{U}$.

We will refine the definition of U^* in section 1.4.

1.1.4 $-^*$ on small types

For function types, we define $-^*$ as follows:

$$\frac{\Gamma.x : A \vdash B : \mathsf{U}}{\Gamma \vdash \Pi(x : A).B : \mathsf{U}}$$

$$\frac{\Gamma.x : A \vdash B : \mathsf{U} \quad \Gamma^{=} \vdash f_0 : (\Pi(x : A).B)[\mathsf{0}] \quad \Gamma^{=} \vdash f_1 : (\Pi(x : A).B)[\mathsf{1}]}{\Gamma^{=} \vdash f_0 \sim_{(\Pi(x:A).B)^*} f_1 \equiv \Pi(x_0 : A[\mathsf{0}], x_1 : A[\mathsf{1}], x_2 : x_0 \sim_{A^*} x_1).f_0\, x_0 \sim_{B^*} f_1\, x_1 : \mathsf{U}}$$

Note that $(\Pi(x : A).B)[i] \equiv \Pi(x : A[i]).B[i_\Gamma, x \mapsto x]$ and $f_0\, x_0 \sim_{B^*} f_1\, x_1$ is in the context $\Gamma^{=}.x_0 : A[\mathsf{0}].x_1 : A[\mathsf{1}].x_2 : x_0 \sim_{A^*} x_1 \equiv (\Gamma.x : A)^{=}$ where Π is the binder which adds the three additional types to the context. In this extended context $f_0\, x_0 : B[\mathsf{0}]$ and $f_1\, x_1 : B[\mathsf{1}]$.

$$\frac{\Gamma.x : A \vdash B : \mathsf{U}}{\Gamma \vdash \Sigma(x : A).B : \mathsf{U}}$$

$$\frac{\Gamma.x : A \vdash B : \mathsf{U} \quad \Gamma^= \vdash w_0 : (\Sigma(x : A).B)[\mathsf{0}] \quad \Gamma^= \vdash w_1 : (\Sigma(x : A).B)[\mathsf{1}]}{\begin{array}{r}\Gamma^= \vdash w_0 \sim_{(\Sigma(x:A).B)^*} w_1 \equiv \Sigma(x_2 : \pi_0\, w_0 \sim_{A^*} \pi_0\, w_1) \\ .\pi_1\, w_0 \sim_{B^*[x_0 \mapsto \pi_0\, w_0, x_1 \mapsto \pi_0\, w_1, x_2 \mapsto x_2]} \pi_1\, w_1 : \mathsf{U}\end{array}}$$

Here we also have $(\Sigma(x : A).B)[i] \equiv \Sigma(x : A[i]).B[i, x \mapsto x]$. As $\sim_{B^*}$ works in context $(\Gamma.x : A)^=$, we need to provide the missing elements of the context i.e. x_0, x_1 and x_2 by a substitution s.t. the arguments $\Gamma^= \vdash \pi_1\, w_i : B[(i_\Gamma, x \mapsto \pi_0\, w_i)]$ still typecheck. We substitute B^* by

$$(\mathsf{id}_{\Gamma^=}, x_0 \mapsto \pi_0\, w_0, x_1 \mapsto \pi_0\, w_1, x_2 \mapsto x_2) : \Gamma^=.x_2 : \pi_0\, w_0 \sim_{A^*} \pi_0\, w_1 \to (\Gamma.x : A)^=$$

which makes the type of the relation $B[\mathsf{0}_\Gamma, x \mapsto \pi_0\, w_0] \to B[\mathsf{1}_\Gamma, x \mapsto \pi_0 w_1] \to \mathsf{U}$.

Another notation for (named) Σ types is the following:

$$\frac{\Gamma \vdash A : \mathsf{U} \quad \Gamma.\mathsf{a} : A \vdash B : \mathsf{U}}{\Gamma \vdash \Sigma(\mathsf{a} : A).(\mathsf{b} : B) : \mathsf{U}}$$

$$\frac{\Gamma.\mathsf{a} : A \vdash B : \mathsf{U} \quad \text{for } i \in \{\mathsf{0}, \mathsf{1}\}\, .\, \Gamma^= \vdash w_i : \Sigma(\mathsf{a} : A[i]).(\mathsf{b} : B[i])}{\Gamma^= \vdash w_0 \sim_{(\Sigma(\mathsf{a}:A).(\mathsf{b}:B))^*} w_1 \equiv \Sigma(\mathsf{a}_2 : \mathsf{a}\, w_0 \sim_{A^*} \mathsf{a}\, w_1).(\mathsf{b}_2 : \mathsf{b}\, w_0 \sim_{B^*} \mathsf{b}\, w_1)}$$

For base types such as $\mathbb{N}$, we define $\sim_{\mathbb{N}^*}$ for all constructors:

$$\frac{}{\Gamma^= \vdash \mathsf{zero} \sim_{\mathbb{N}^*} \mathsf{zero} \equiv \mathsf{1}} \qquad \frac{\Gamma^= \vdash m : \mathbb{N}}{\Gamma^= \vdash \mathsf{suc}\, m \sim_{\mathbb{N}^*} \mathsf{zero} \equiv \mathsf{0}} \qquad \frac{\Gamma^= \vdash n : \mathbb{N}}{\Gamma^= \vdash \mathsf{zero} \sim_{\mathbb{N}^*} \mathsf{suc}\, n \equiv \mathsf{0}}$$

$$\frac{\Gamma^= \vdash m, n : \mathbb{N}}{\Gamma^= \vdash \mathsf{suc}\, m \sim_{\mathbb{N}^*} \mathsf{suc}\, n \equiv m \sim_{\mathbb{N}^*} n}$$

1.1.5 $-^*$ on terms

$$\frac{\Gamma.x : A \vdash b : B}{\Gamma \vdash \lambda x.b : \Pi(x : A).B} \qquad \frac{\Gamma.x : A \vdash b : B}{\Gamma^= \vdash (\lambda x.b)^* \equiv \lambda \underbrace{x_0}_{:A[\mathsf{0}]}, \underbrace{x_1}_{:A[\mathsf{1}]}, \underbrace{x_2}_{:x_0 \sim_{A^*} x_1}. \underbrace{b^*}_{\substack{:b[\mathsf{0}] \sim_{B^*} b[\mathsf{1}] \\ \equiv (\lambda x.b)[\mathsf{0}]\, x_0 \sim_{B^*} (\lambda x.b)[\mathsf{1}]\, x_1}}}$$

$$\frac{\Gamma \vdash f : \Pi(x : A).B \quad \Gamma \vdash u : A}{\Gamma \vdash f\, u : B[x \mapsto u]} \qquad \frac{\Gamma \vdash f : \Pi(x : A).B \quad \Gamma \vdash u : A}{\Gamma^= \vdash (f\, u)^* \equiv \underbrace{\underbrace{f^*}_{:f[\mathsf{0}] \sim_{(\Pi(x:A).B)^*} f[\mathsf{1}]} \; u[\mathsf{0}]\, u[\mathsf{1}] \; \underbrace{u^*}_{u[\mathsf{0}] \sim_{A^*} u[\mathsf{1}]}}_{:f[\mathsf{0}]\, u[\mathsf{0}] \sim_{(B[x \mapsto u])^*} f[\mathsf{1}]\, u[\mathsf{1}]}}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[x \mapsto a]}{\Gamma \vdash (a, b) : \Sigma(x : A).B}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[x \mapsto a]}{\Gamma^= \vdash (a, b)^* \equiv (\underbrace{a^*}_{\substack{:a[\mathsf{0}] \sim_{A^*} a[\mathsf{1}] \\ \equiv \pi_0\, (a,b)[\mathsf{0}] \sim_{A^*} \pi_0\, (a,b)[\mathsf{1}]}}, \underbrace{b^*}_{\substack{:b[\mathsf{0}] \sim_{(B[x \mapsto a])^*} b[\mathsf{1}] \\ \equiv \pi_1\, (a,b)[\mathsf{0}] \sim_{B^*[x_0 \mapsto a[\mathsf{0}], x_1 \mapsto a[\mathsf{1}], x_2 \mapsto a^*]} \pi_1\, (a,b)[\mathsf{1}]}})}$$

$$\frac{\Gamma \vdash w : \Sigma(x:A).B}{\Gamma \vdash \pi_0\, w : A} \qquad \frac{\Gamma \vdash w : \Sigma(x:A).B}{\Gamma^= \vdash (\pi_0\, w)^* \equiv \pi_0\,(w^*) : \underbrace{\pi_0\, w[0] \sim_{A^*} \pi_0\, w[1]}_{\equiv (\pi_0\, w)[0] \sim_{A^*} (\pi_0\, w)[1]}}$$

$$\frac{\Gamma \vdash w : \Sigma(x:A).B}{\Gamma \vdash \pi_1\, w : B[x \mapsto \pi_0\, w]}$$

$$\frac{\Gamma \vdash w : \Sigma(x:A).B}{\Gamma^= \vdash (\pi_1\, w)^* \equiv \pi_1\,(w^*) : \underbrace{(\pi_1\, w[0] \sim_{B^*[x_0 \mapsto \pi_0\, w[0], x_1 \mapsto \pi_0\, w[1], x_2 \mapsto \pi_0\, w^*]} \pi_1\, w[1])}_{\equiv (\pi_1\, w)[0] \sim_{(B[x \mapsto \pi_0\, w])^*} (\pi_1\, w)[1]}}$$

If we projected from a named Σ-type, eg. $\Sigma(\mathsf{a} : A).(\mathsf{b} : B)$, we have $(\mathsf{a}\, w)^* \equiv \mathsf{a}_2\, w^*$ and similarly for b.

Variables:

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma.x : A \vdash x : A} \qquad \frac{\Gamma \vdash A : \mathsf{U}}{(\Gamma.x : A)^= \vdash x^* \equiv x_2 : x_0 \sim_{A^*} x_1}$$

Base types, eg. natural numbers:

$$\Gamma^= \vdash \mathsf{zero}^* \equiv * : \overbrace{\mathsf{zero} \sim_{\mathbb{N}^*} \mathsf{zero}}^{\equiv 1}$$

$$\frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathsf{suc}\, n : \mathbb{N}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma^= \vdash (\mathsf{suc}\, n)^* \equiv n^* : \underbrace{\mathsf{suc}\, n[0] \sim_{\mathbb{N}^*} \mathsf{suc}\, n[1]}_{\equiv n[0] \sim_{\mathbb{N}^*} n[1]}}$$

$$\frac{\Gamma \vdash P : \mathbb{N} \to \mathsf{U} \quad \Gamma \vdash c : P\, \mathsf{zero} \quad \Gamma \vdash d : \Pi(n : \mathbb{N}).P\, n \to P\,(\mathsf{suc}\, n)}{\Gamma \vdash \mathsf{ind}_{\mathbb{N}}\, P\, c\, d : \Pi(n : \mathbb{N}).P\, n}$$

$$\frac{\Gamma \vdash P : \mathbb{N} \to \mathsf{U} \quad \Gamma \vdash c : P\, \mathsf{zero} \quad \Gamma \vdash d : \Pi(n : \mathbb{N}).P\, n \to P\,(\mathsf{suc}\, n)}{\begin{array}{l} \Gamma^= \vdash (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)^* : \Pi(n : \mathbb{N})^=.(\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)[0_\Gamma]\, n_0 \sim_{(P\, n)^*} (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)[1_\Gamma]\, n_1 \\ \Gamma^= \vdash (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)^*\, \mathsf{zero}\, \mathsf{zero}\, \_ \equiv c^* : c[0] \sim_{(P\, \mathsf{zero})^*} c[1] \\ \Gamma^= \vdash (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)^*\, \mathsf{zero}\, (\mathsf{suc}\, \_)\, r \equiv \mathsf{ind}_0\, r \\ \Gamma^= \vdash (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)^*\, (\mathsf{suc}\, \_)\, \mathsf{zero}\, r \equiv \mathsf{ind}_0\, r \\ \Gamma^= \vdash (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d)^*\, (\mathsf{suc}\, n_0)\, (\mathsf{suc}\, n_1)\, n_2 \equiv (d\, n\, (\mathsf{ind}_{\mathbb{N}}\, P\, c\, d))^* \\ \qquad : d[0]\, n_0\, (\mathsf{ind}_{\mathbb{N}}\, P[0]\, c[0]\, d[0]\, n_0) \sim_{(P\,(\mathsf{suc}\, n))^*} d[1]\, n_1\, (\mathsf{ind}_{\mathbb{N}}\, P[1]\, c[1]\, d[1]\, n_1) \end{array}}$$

1.1.6 $-^=$ on substitutions

We extend the definition of $-^=$ from terms to substitutions to justify the following rule:

$$\frac{\rho : \Delta \to \Gamma}{\rho^= : \Delta^= \to \Gamma^=}$$

We define $\rho^=$ by induction on the cardinality of the codomain[1]:

$$\frac{\Gamma \vdash}{\Gamma \vdash () : \emptyset} \qquad \frac{\Gamma \vdash}{\Gamma^= \vdash ()^= \equiv () : \emptyset}$$

[1]For the last rule to typecheck, we use equation 6 when extending the substitution $\rho^=$ as we expect $u[0]$ to have type $A[0][\delta^=]$, but it has type $A[\delta 0]$). Also note that when extending the substitution by $u^= : u[0] \sim_{(A[\rho])^*} u[1]$, we need something of type $(x_0 \sim_{A^*} x_1)[(\rho^=, x_0 \mapsto u[0], x_1 \mapsto u[1])]$ but this is equal to the former.

$$\frac{\rho : \Delta \to \Gamma \quad \Gamma \vdash A \quad \Delta \vdash u : A[\rho]}{(\rho, x \mapsto u) : \Delta \to \Gamma.x : A}$$

$$\frac{\rho : \Delta \to \Gamma \quad \Gamma \vdash A \quad \Delta \vdash u : A[\rho]}{(\rho, x \mapsto u)^= \equiv (\rho^=, x_0 \mapsto u[0_\Delta], x_1 \mapsto u[1_\Delta], x_2 \mapsto u^*) : \Delta^= \to (\Gamma.x : A)^=}$$

Now we can define how $-^*$ acts on substituted terms:

$$(a[\rho])^* \equiv a^*[\rho^=].$$

$-^=$ is functorial on substitutions:

$$(\rho\sigma)^= \equiv \rho^=\sigma^= \tag{4}$$

$$\mathsf{id}_\Gamma^= \equiv \mathsf{id}_{\Gamma^=} \tag{5}$$

Lifted substitutions obey a naturality rule:

$$\frac{\rho : \Delta \to \Gamma}{i_\Gamma \rho^= \equiv \rho i_\Delta} \; i \in \{0, 1\} \tag{6}$$

Note that $i_{\Gamma^=}$ is not the same as $(i_\Gamma)^=$, this can be checked by taking Γ to be a single-element context.

1.2 Higher dimensions

An element of the context $x : A$ can be viewed as a point of type A. An element of the context $(x : A)^=$ is a line x_2 with edges x_0 and x_1 (3 elements). Note that this is a heterogeneous line: the types of points are $A[0]$ and $A[1]$ which can differ if A is not a closed type.

More generally, an element of the context $(x : A)^n$ ($-^=$ iterated n times) has 3^n components which make an n-dimensional cube. The first three iterations:

$$\begin{array}{lll}
\Gamma & \vdash x : A & \\
\Gamma^= & \vdash (x : A)^= & \equiv \quad x_0 : A[0_\Gamma].x_1 : A[1_\Gamma].x_2 : x_0 \sim_{A^*} x_1 \\
\Gamma^2 & \vdash (x : A)^2 & \equiv
\end{array}$$

$$\begin{array}{lll}
x_{00} : A[0_\Gamma 0_{\Gamma^=}] & .x_{01} : A[0_\Gamma 1_{\Gamma^=}] & .x_{02} : x_{00} \sim_{A[0_\Gamma]^*} x_{01} \\
.x_{10} : A[1_\Gamma 0_{\Gamma^=}] & .x_{11} : A[1_\Gamma 1_{\Gamma^=}] & .x_{12} : x_{10} \sim_{A[1_\Gamma]^*} x_{11} \\
.x_{20} : x_{00} \sim_{A^*[0_{\Gamma^=}]} x_{10} & .x_{21} : x_{01} \sim_{A^*[1_{\Gamma^=}]} x_{11} & .x_{22} : x_{20} \sim_{(x_0 \sim_{A^*} x_1)^*} x_{21}
\end{array}$$

Note that the type of x_{20} was computed from $(x_0 \sim_{A^*} x_1)[0_{\Gamma^=.x_0:A[0].x_1:A[1]}]$. Also, the type of x_{22} is equal to $A^{**}\, x_{00}\, x_{01}\, x_{02}\, x_{10}\, x_{11}\, x_{12}\, x_{20}\, x_{21}$. A^{**} is a relation defining a square (a 2-dimensional relation). This relation has 8 arguments, the 4 points and 4 sides of the square.

$\Gamma^3 \vdash (x : A)^3 \equiv$

$x_{000} : A[000]$	$.x_{001} : A[001]$	$.x_{002} : x_{000} \sim_{A[00]^*} x_{001}$
$.x_{010} : A[010]$	$.x_{011} : A[011]$	$.x_{012} : x_{010} \sim_{A[01]^*} x_{011}$
$.x_{020} : x_{000} \sim_{A[0]^*[0]} x_{010}$	$.x_{021} : x_{001} \sim_{A[0]^*[1]} x_{011}$	$.x_{022} : x_{020} \sim_{(x_{00} \sim_{A[0]^*} x_{01})^*} x_{021}$
$.x_{100} : A[100]$	$.x_{101} : A[101]$	$.x_{102} : x_{100} \sim_{A[10]^*} x_{101}$
$.x_{110} : A[110]$	$.x_{111} : A[111]$	$.x_{112} : x_{110} \sim_{A[11]^*} x_{111}$
$.x_{120} : x_{100} \sim_{A[1]^*[0]} x_{110}$	$.x_{121} : x_{101} \sim_{A[1]^*[1]} x_{111}$	$.x_{122} : x_{120} \sim_{(x_{10} \sim_{A[1]^*} x_{11})^*} x_{121}$
$.x_{200} : x_{000} \sim_{A^*[00]} x_{100}$	$.x_{201} : x_{001} \sim_{A^*[01]} x_{101}$	$.x_{202} : x_{200} \sim_{(x_{00} \sim_{A^*[0]} x_{10})^*} x_{201}$
$.x_{210} : x_{010} \sim_{A^*[10]} x_{110}$	$.x_{211} : x_{011} \sim_{A^*[11]} x_{111}$	$.x_{212} : x_{210} \sim_{(x_{01} \sim_{A^*[1]} x_{11})^*} x_{211}$
$.x_{220} : x_{200} \sim_{(x_0 \sim_{A^*} x_1)^*[0]} x_{210}$	$.x_{221} : x_{201} \sim_{(x_0 \sim_{A^*} x_1)^*[1]} x_{211}$	$.x_{222} : x_{220} \sim_{(x_{20} \sim_{(x_0 \sim_{A^*} x_1)^*} x_{21})^*} x_{221}$

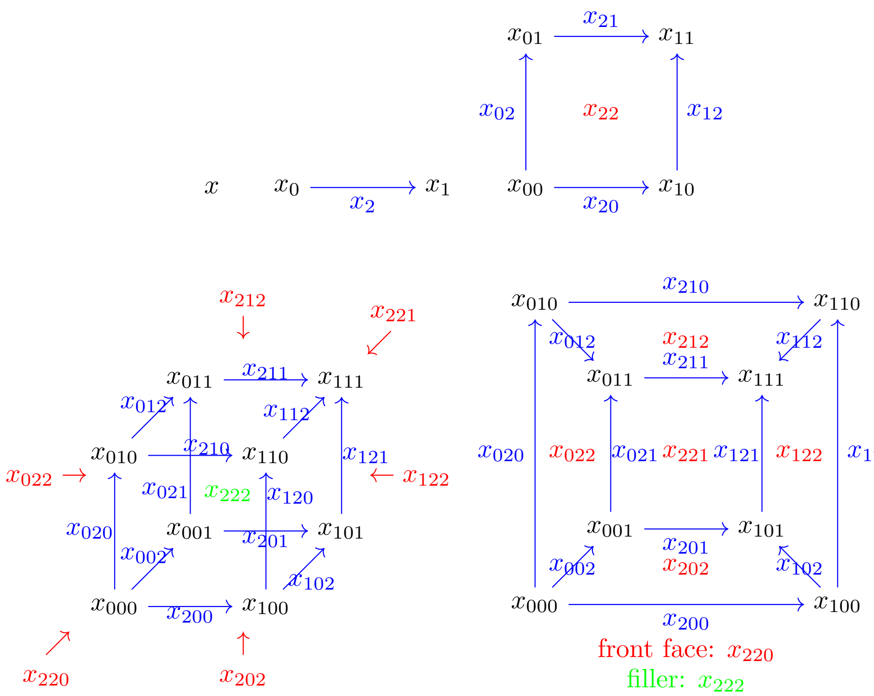


Figure 1: Cubes of dimension 0-3. The 3-dimensional cube is shown twice to explain the second depiction which we will use later.

Generally, an element of a context $(x : A)^n$ corresponds to an n-dimensional cube (cube of level n), we name it's dimensions $d_0, d_1, ..., d_{n-1}$. A variable x_{ijkl} refers to the face of a 4-dimensional cube which has coordinate i in dimension d_0, coordinate j in dimension d_1, coordinate k in dimension d_2 and coordinate l in dimension d_3. By face we mean any 0,1,2,3 or 4-dimensional face, a 4-dimensional cube only has one 4-dimensional face, itself. A coordinate can be 0 or 1, in this case we can interpret it as a usual cartesian coordinate. If the coordinate is 2, it means that this is a face spanning through that dimension.

Eg. x_{011} is the $d_0 = 0, d_1 = 1, d_2 = 1$ point of a 3-dimensional cube (left upper back if the dimensions are oriented in the standard way); x_{021} is a line only in the d_1 dimension, so it is the line on the left of the top of the cube; x_{221} is the back face of the cube; x_{222} is the cube itself (the filler).

1.2.1 Notation for repeated application of $-^=$

We will denote the n-times repeated application of $-^=$ by $-^n$ and generalise rule 1 as:

$$\frac{\Gamma \vdash u : A}{\Gamma^= \vdash u^= : A^=} \qquad \rightarrow \qquad \frac{\Gamma \vdash u : A}{\Gamma^n \vdash u^n : A^n}$$

Definitions:

$$\Gamma^0 \equiv \Gamma \qquad \Gamma^{n+1} \equiv (\Gamma^n)^=$$

$$\rho^0 \equiv \rho \qquad \rho^{n+1} \equiv (\rho^n)^=$$

$$\frac{\Gamma \vdash a : A}{\Gamma^0 \vdash a^0 \equiv a : A^0} \qquad \frac{\Gamma \vdash a : A}{\Gamma^{n+1} \vdash a^{n+1} \equiv (a^n)^= : A^{n+1}}$$

In the last rule, a^n is the last 3^n terms in the substitution $(\mathsf{id}_{\Gamma^n}, a^n) : \Gamma^n \to \Gamma^n.A^n$, so applying $-^=$ on it results in the substitution $(\mathsf{id}_{\Gamma^n}, a^n)^= : \Gamma^{n+1} \to \Gamma^{n+1}.A^{n+1}$, in which the last 3^{n+1} terms are written[2] as $(a^n)^=$.

We also define the context $(\Gamma.x : A)_n \subset (\Gamma.x : A)^n$ where the last element (the filler) is removed, and the type $(x : A)^{*n}$ which is the last element, so we have

$$(\Gamma.x : A)^n \equiv \Gamma^n.(x : A)_n.x_{2^n} : (x : A)^{*n}.$$

x_{2^n} is a meta-notation for $x_{\underbrace{22\ldots2}_{n \text{ times}}}$.

We define $(x : A)_n$ and $(x : A)^{*n}$ by induction on n:

$$\begin{aligned}
&\Gamma^0 && \vdash (x : A)_0 && \equiv \emptyset \\
&\Gamma^{n+1} && \vdash (x : A)_{n+1} && \equiv ((x : A)_n)^=.x_{2^n0} : (x : A)^{*n}[0_{\Gamma^n.(x:A)_n}] \\
&&&&& \quad .x_{2^n1} : (x : A)^{*n}[1_{\Gamma^n.(x:A)_n}] \\
&\Gamma^0.(x : A)_0 && \vdash (x : A)^{*0} && \equiv A \\
&\Gamma^{n+1}.(x : A)_{n+1} && \vdash (x : A)^{*n+1} && \equiv x_{2^n0} \sim_{((x:A)^{*n})^*} x_{2^n1}
\end{aligned}$$

The following derivation helps understanding the last definition:

$$\cfrac{\cfrac{\Gamma^n.(x : A)_n \vdash (x : A)^{*n} : \mathsf{U}}{(\Gamma^n.(x : A)_n)^= \vdash ((x : A)^{*n})^* : (x : A)^{*n}[0] \sim_{\mathsf{U}^*} (x : A)^{*n}[1]}}{(\Gamma^n.(x : A)_n)^= \vdash \sim_{((x:A)^{*n})^*} : (x : A)^{*n}[0] \to (x : A)^{*n}[1] \to \mathsf{U}}$$

$A_n^= \subset A^{n+1}$ is the context where the last three elements are omitted.

For any term u we define the two parts of the substitution u^n corresponding to the above contexts:

$$\frac{\Gamma \vdash u : A}{\Gamma^n \vdash u^n \equiv (u_n, u^{*n}) : A^n} \qquad \frac{\Gamma \vdash u : A}{\Gamma^n \vdash (u)_n : A_n} \qquad \frac{\Gamma \vdash u : A}{\Gamma^n \vdash u^{*n} : (x : A)^{*n}[(u)_n]}$$

We define $(u)_n$ and u^{*n} by induction on n. Given a $\Gamma \vdash u : A$,

$$\begin{aligned}
&\Gamma^0 \vdash (u)_0 \equiv () && : A_0 \qquad && \Gamma^0 \vdash u^{*0} \equiv u && : (x : A)^{*0} \\
&\Gamma^{n+1} \vdash (u)_{n+1} \equiv ((u)_n^=, u^{*n}[0], u^{*n}[1]) && : A_{n+1} \qquad && \Gamma^{n+1} \vdash u^{*n+1} \equiv (u^{*n})^* && : (x : A)^{*n+1}[(u)_{n+1}]
\end{aligned}$$

[2] Note that $-^*$ is not defined on substitutions, so $(a^n)^=$ is not the same as $(a^n[0], a^n[1], (a^n)^*)$; in fact, the last element of this context does not make sense.

Note that

$$\Gamma^{n+1}.(x:A)^{=}_{n} \vdash ((x:A)^{*n})^{*} \equiv A^{*n+1}\,(x)^{=}_{n} : (x:A)^{*n}[\mathsf{0}] \sim_{\mathsf{U}^*} (x:A)^{*n}[\mathsf{1}].$$

Also, if $\sim$ is the identity function i.e. until we redefine U^* in section 1.4.1, we have

$$\Gamma^{n}.(x:A)_{n} \vdash (x:A)^{*n} \equiv A^{*n}\,(x)_{n} : \mathsf{U}.$$

1.2.2 Type formers in higher dimensions

For any type A, the last element of the context $(x:A)^{n}$ is given by $(x:A)^{*n}$ and is typed by the following rule:

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^{n}(x:A)_{n} \vdash (x:A)^{*n} : \mathsf{U}}$$

For the different type formers resulting in A we can describe $(x:A)^{*n}$ uniformly (note that the first two definitions will change after redefining U^* in section 1.4.1):

$$\begin{aligned}
&(X:\mathsf{U})_{n} && \vdash (X:\mathsf{U})^{*n} && \equiv (x:X)_{n} \to \mathsf{U}\\
&(X:\mathsf{U})^{n}.(x:X)_{n} && \vdash (x:X)^{*n} && \equiv X_{2^n}\,(x)_{n}\\
&(f:\Pi(x:A).B)_{n} && \vdash (f:\Pi(x:A).B)^{*n} && \equiv \Pi(x:A)^{n}.(f\,x:B)^{*n}\\
& && && \equiv \Pi(x:A)^{n}.(y:B)^{*n}[(f\,x)_{n}]\\
&(w:\Sigma(x:A).B)_{n} && \vdash (w:\Sigma(x:A).B)^{*n} && \equiv \Sigma(r:(x:A)^{*n}[(\pi_0\,w)_{n}])\\
& && && \quad .(y:B)^{*n}[(\pi_0\,w)_{n}, r, (\pi_1\,w)_{n}]
\end{aligned}$$

The above notations are not new syntax, they can be seen as macros, e.g. $\Pi(x:A)^{n}$ means the binder Π iterated 3^{n}-times.

Using the above insight, we can interpret higher types for each individual type former as follows.

- An element of $(X:\mathsf{U})^{n}$ is a type of heterogeneous n-dimensional cubes. Eg. an element of $(X:\mathsf{U})^{2}$ consists of four types of points, four types of lines and a type of squares.

- An element of $(\Pi(x:A).B)^{n}$ can be viewed as a (pointwise) function from $(x:A)^{n}$ to $(y:B)^{n}$, where the action on the last element of $(x:A)^{n}$ is compatible with the action on the lower elements.

$$(\Pi(x:A).B)^{n} \simeq \Pi(x:A)^{n}.(y:B)^{n}$$

- $(\Sigma(x:A).B)^{n}$ is just a dependent pair of an $(x:A)^{n}$ and $(y:B)^{n}$.

$$(\Sigma(x:A).B)^{n} \simeq \Sigma(x:A)^{n}.(y:B)^{n}$$

For base types such as $\mathbb{N}$, $(x:\mathbb{N})^{*n}$ is defined recursively and reduces to $\mathsf{0}$ or $\mathsf{1}$, once all the elements of $(x:\mathbb{N})_{n}$ have been substituted in the empty context. (TODO: This is not true for higher inductive types.)

1.3 Homogeneous equality

This section can be seen as a different presentation of the internalisation of parametricity [2]. However we use binary relations instead of unary relations.

An external logical relation can be defined by induction over the syntax, as we did it in the previous section, but this relation lives in a different context, $\Gamma^=$ instead of Γ:

$$\frac{\Gamma \vdash A}{\Gamma^= \vdash \sim_{A^*} : A[0] \to A[1] \to \mathsf{U}}$$

To internalize this relation i.e. to have it in the same context we define the substitution R and the term refl by adding the following rules:

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathsf{R}_\Gamma : \Gamma^=} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}\, a \equiv a^*[\mathsf{R}_\Gamma] : a \sim_{A^*[\mathsf{R}_\Gamma]} a}$$

$$\begin{aligned} \mathsf{R}_\emptyset &\equiv () && : \emptyset \to \emptyset \\ \mathsf{R}_{\Gamma.x:A} &\equiv (\mathsf{R}_\Gamma, x, x, \mathsf{refl}\, x) && : \Gamma.x : A \to (\Gamma.A)^= \end{aligned}$$

Sometimes we write r for refl and $\mathsf{R}_{\Gamma.x}$ for $\mathsf{R}_{\Gamma.x:A}$.

We also add the substitution law

$$\frac{\Delta \vdash \rho : \Gamma \quad \Gamma \vdash a : A}{\Delta \vdash (\mathsf{refl}\, a)[\rho] \equiv \mathsf{refl}\,(a[\rho]).}$$

R has the property

$$i_\Gamma \mathsf{R}_\Gamma \equiv \mathsf{id}_\Gamma \tag{7}$$

for $i \in \{0, 1\}$, so by substituting with R, $\Gamma^= \vdash \sim_{A^*} : A[0] \to A[1] \to \mathsf{U}$ becomes $\Gamma \vdash =_A \equiv \sim_{A^*[\mathsf{R}_\Gamma]} \equiv \sim_{\mathsf{refl}\, A} : A \to A \to \mathsf{U}$, this is how we define the usual homogeneous equality.

Now we can define the operations ap and apd simply by refl:

$$\frac{\Gamma \vdash A, B : \mathsf{U} \quad \Gamma \vdash f : A \to B \quad \Gamma \vdash a, a' : A \quad \Gamma \vdash p : a \sim_{A^*[\mathsf{R}_\Gamma]} a'}{\Gamma \vdash \mathsf{ap}\, f\, a\, a'\, p \equiv \underbrace{f^*[\mathsf{R}_\Gamma]}_{\equiv \mathsf{refl}\, f}\, a\, a'\, p : \underbrace{f\, a \sim_{B^*[\mathsf{R}_\Gamma]} f\, a'}_{\equiv f\, a =_B f\, a'}}$$

$$\frac{\Gamma \vdash A : \mathsf{U} \quad \Gamma.x : A \vdash B : \mathsf{U} \quad \Gamma \vdash f : \Pi(x : A).B \quad \Gamma \vdash a, a' : A \quad \Gamma \vdash p : a \sim_{A^*[\mathsf{R}_\Gamma]} a'}{\Gamma \vdash \mathsf{apd}\, f\, a\, a'\, p \equiv \underbrace{f^*[\mathsf{R}_\Gamma]}_{\equiv \mathsf{refl}\, f}\, a\, a'\, p : \underbrace{f\, a \sim_{B^*[\mathsf{R}_\Gamma, x_0 \mapsto a, x_1 \mapsto a', x_2 \mapsto p]} f\, a'}_{\not\equiv f\, a =_B f\, a'}}$$

Note how the heterogeneous equality in the target of apd is expressed.

R also obeys the following rule:

$$\frac{\rho : \Delta \to \Gamma}{\rho^= \mathsf{R}_\Delta \equiv \mathsf{R}_\Gamma \rho : \Delta \to \Gamma^=} \tag{8}$$

By adding refl we added a new normal form to our type theory, $\mathsf{refl}\, x$ reduces to itself if x is a variable:

$$x : A \vdash \mathsf{refl}\, x \equiv x_2[\mathsf{R}_{x:A}] \equiv x^*[x_0 \mapsto x, x_1 \mapsto x, x_2 \mapsto \mathsf{refl}\, x] \equiv \mathsf{refl}\, x$$

This poses the questions what $(\mathsf{refl}\, x)^*$, $(\mathsf{refl}\, x)^{**}$ etc. should reduce to. We give the answer in section 1.3.1.

The construction for the internalisation of parametricity in [2] uses the notation $[\![-]\!]$ for refl, and these semantic brackets are decorated with a list of bound variables ξ, which in our construction corresponds to the $-^{=}$-d part of the context. Their rule

$$\frac{\Gamma.\xi \vdash t : A}{\Gamma.\xi^{=} \vdash [\![t]\!]_\xi : [\![A]\!]_\xi \, (t[\mathsf{0}_\xi])\, (t[\mathsf{1}_\xi])}$$

(which was extended to the binary case) corresponds to

$$\frac{\Gamma.\xi \vdash t : A}{\Gamma.\xi^{=} \vdash t^*[\mathsf{R}_\Gamma] : t[\mathsf{0}_\xi] \sim_{A^*} [\mathsf{R}_\Gamma]\, t[\mathsf{1}_\xi]}$$

in our notation. Note that the context $\Gamma.\xi^{=}$ only makes sense if ξ does not depend on Γ.

1.3.1 $-^*$ on refl

So far we have defined the action of $-^*$ on every term except on refl. Perhaps a natural definiton would be $(\mathsf{refl}\, x)^* \equiv \mathsf{refl}\,(x^*)$, however these expressions have different types (given a closed type A):

$$\begin{aligned}
(x : A)^{=} &\vdash (\mathsf{refl}\, x)^* : \mathsf{refl}\, x_0 \sim_{(x \sim_{\mathsf{refl}\, A}\, x)^*} \mathsf{refl}\, x_1 \equiv A^{**}\, x_0\, x_1\, x_2\, x_0\, x_1\, x_2\, (\mathsf{refl}\, x_0)\, (\mathsf{refl}\, x_1)\\
(x : A)^{=} &\vdash \mathsf{refl}\, x^* \quad : x_2 \sim_{\mathsf{refl}\, (x_0 \sim_{A^*} x_1)} x_2 \qquad \equiv A^{**}\, x_0\, x_0\, (\mathsf{refl}\, x_0)\, x_1\, x_1\, (\mathsf{refl}\, x_1)\, x_2\, x_2
\end{aligned}$$

The order of arguments of A^{**} differs: in fact, if we visualize the arguments as points and lines of a square we realize that one can be derived from the other by swapping the dimensions of the square - the two different refls at dimension 2 (refl and refl^*) correspond to the two different ways of creating a square from a line:

$$\begin{tikzcd}
x_1 \arrow[r, "\mathsf{refl}\, x_1"] & x_1 \\
x_0 \arrow[u, "x_2"] \arrow[r, "\mathsf{refl}\, x_0"'] \arrow[ur, phantom, "{\begin{matrix}(\mathsf{refl}\, x)^* \\ \equiv \mathsf{refl}^*\, x_0\, x_1\, x_2\end{matrix}}"] & x_0 \arrow[u, "x_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "x_2"] & x_1 \\
x_0 \arrow[u, "\mathsf{refl}\, x_0"] \arrow[r, "x_2"'] \arrow[ur, phantom, "{\begin{matrix}\mathsf{refl}\, x^* \\ \equiv \mathsf{refl}\, x_2\end{matrix}}"] & x_1 \arrow[u, "\mathsf{refl}\, x_1"']
\end{tikzcd}$$

This motivates the definition of the substitution S which swaps the two dimensions:

$$\begin{aligned}
\mathsf{S}_\emptyset \quad &\equiv () && : \emptyset \to \emptyset\\
\mathsf{S}_{\Gamma.x:A} &\equiv (\mathsf{S}_\Gamma, x_{00} \mapsto x_{00}, x_{01} \mapsto x_{10}, x_{02} \mapsto x_{20},\\
&\qquad x_{10} \mapsto x_{01}, x_{11} \mapsto x_{11}, x_{10} \mapsto x_{01},\\
&\qquad x_{20} \mapsto x_{02}, x_{21} \mapsto x_{12}, x_{22} \mapsto x_{22}[\mathsf{S}_{\Gamma.x:A}]) && : (\Gamma.x : A)^2 \to (\Gamma.x : A)^2
\end{aligned}$$

Visually:

$$\begin{tikzcd}
x_{01} \arrow[r, "x_{21}"] & x_{11} \\
x_{00} \arrow[u, "x_{02}"] \arrow[r, "x_{20}"'] \arrow[ur, phantom, "x_{22}"] & x_{10} \arrow[u, "x_{12}"']
\end{tikzcd}
\xmapsto{\ \mathsf{S}_{\Gamma.x:A}\ }
\begin{tikzcd}
x_{10} \arrow[r, "x_{12}"] & x_{11} \\
x_{00} \arrow[u, "x_{20}"] \arrow[r, "x_{02}"'] \arrow[ur, phantom, "{x_{22}[\mathsf{S}_{\Gamma.x:A}]}"] & x_{01} \arrow[u, "x_{21}"']
\end{tikzcd}$$

Just as $\mathsf{refl}\, x \equiv x_2[\mathsf{R}]$ was a normal form if x was a variable, $x_{22}[\mathsf{S}]$ becomes a new normal form for swapping.

Now we define $(\mathsf{R}_\Gamma)^=$ as $\mathsf{S}_\Gamma \mathsf{R}_{\Gamma^=}$. But as a consequence of this, we need to take care of higher swaps like $(\mathsf{S}_{\Gamma^=})^= : \Gamma^4 \to \Gamma^4$ etc.

Given $i + j = n - 2$, $\mathsf{S}^j_{\Gamma^i}$ is an n-dimensional swap substitution. We map this substitution to the n-dimensional permutation which swaps the i^{th} and $(i+1)^{\text{th}}$ element. For example, $\mathsf{S}^1_{\Gamma^2}$ is mapped to 01324, $\mathsf{S}^0_{\Gamma^3}$ is mapped to 01243. When composing two such substitutions, we compose the corresponding permutations, and if they result in the identity permutation, the composition of the substitutions is definitionally equal to the identity substitution. We say that $x_{2^n}[\rho]$ is a normal form if $x \in \Gamma$ and ρ is a non-identity composition of n-dimensional swaps, followed by R substitutions which are not $-^=$-d:

$$x_{2^n}[\mathsf{S}^{n-2-i_1}_{\Gamma^{i_1}} \mathsf{S}^{n-2-i_2}_{\Gamma^{i_2}} ... \mathsf{S}^{n-2-i_m}_{\Gamma^{i_m}} \mathsf{R}_{\Gamma^{n+1}} \mathsf{R}_{\Gamma^n} ... \mathsf{R}_{\Gamma^{n+1-k}}]$$

where $n \geq 2$, $m \geq 0$, $k \geq -1$. Note that, by the naturality rule 8 we can move Ss to the left hand side: $\mathsf{R}_{\Gamma^{n+2}} \mathsf{S}^{n-i}_{\Gamma^i} \equiv (\mathsf{S}^{n-i}_{\Gamma^i})^= R_{\Gamma^{n+2}} \equiv \mathsf{S}^{n+1-i}_{\Gamma^i} R_{\Gamma^{n+2}}$.

1.3.2 Compute swapping

In the first step of the following derivation, x_{22} is in normal form, in the second step $x_{22}[\mathsf{S}_{\Gamma.x:A}]$ is still in normal form, however in the last step $x_{22}[\mathsf{S}_{\Gamma.x:A}\rho]$ might not be in normal form and it is stuck as we don't have a general rule which makes a swap substitution commute with other substitutions. And ρ might give a value to the variable x_{22} which we could use to make further computations.

$$\frac{\dfrac{(\Gamma.x : A)^2 \vdash x_{22} : (x : A)^{*2}}{(\Gamma.x : A)^2 \vdash x_{22}[\mathsf{S}_{\Gamma.x:A}] : (x : A)^{*2}[\mathsf{S}_{\Gamma.x:A}]} \quad \rho : \Delta \to (\Gamma.x : A)^2}{\Delta \vdash x_{22}[\mathsf{S}_{\Gamma.x:A}\rho] : (x : A)^{*2}[\mathsf{S}_{\Gamma.x:A}\rho]}$$

Note that if the variable is not x_{22} but eg. x_{12}, we are able to make further computation, as $x_{12}[\mathsf{S}_{\Gamma.x:A}] \equiv x_{21}$.

To fix this, we add rules on how to compute $x_{22}[\mathsf{S}_{\Gamma.x:A}]$ depending on the type A.

More generally, we are in the following situation ($n \geq 2, i + j = n - 2$):

$$\frac{(\Gamma.x : A)^n \vdash x_{2^n} : (x : A)^{*n}}{(\Gamma.x : A)^n \vdash x_{2^n}[\mathsf{S}^i_{(\Gamma.x:A)^j}] : (x : A)^{*n}[\mathsf{S}^i_{(\Gamma.x:A)^j}]}$$

If we apply a substitution $\rho : \Delta \to (\Gamma.x : A)^n$ to the term in the conclusion, it does not generally commute with S, so we can't look up x_{2^n}.

However, with the help of the descriptions of higher types given in section 1.2.2, depending on what type A is, we add rules on how to reduce the term $x_{2^n}[\mathsf{S}]$, which allows the lookup of substitutions. If $A \equiv \mathsf{U}$:

$$\begin{array}{lll}
(\Gamma.X : \mathsf{U})^n \vdash & X_{2^n}[\mathsf{S}^i_{(\Gamma.X:\mathsf{U})^j}] & : (X : \mathsf{U})^{*n}[\mathsf{S}^i_{(\Gamma.X:\mathsf{U})^j}] \\
\text{(η-expansion)} & \equiv \lambda(x : X)_n[\mathsf{S}^i_{\Gamma^j}].X_{2^n}[\mathsf{S}^i_{(\Gamma.X:\mathsf{U})^j}]\,(x)_n[\mathsf{S}^i_{(\Gamma.X:\mathsf{U})^j}] & : (x : X)_n[\mathsf{S}^i_{(\Gamma.X:\mathsf{U})^j}] \to \mathsf{U} \\
\text{(new rule)} & \equiv \lambda(x : X)_n[\mathsf{S}^i_{\Gamma^j}].X_{2^n}\,(x)_n[\mathsf{S}^i_{(\Gamma.X:\mathsf{U}.x:X)^j} \mathsf{S}^i_{(\Gamma.X:\mathsf{U})^j}] &
\end{array}$$

Now applying the substitution ρ, X_{2^n} is looked up directly from the context.

If $A \equiv X$ (the type is a variable):

$$\begin{aligned}
(\Gamma.X : \mathsf{U}.x : X)^n \vdash \quad & x_{2^n}[\mathsf{S}^i_{(\Gamma.X:\mathsf{U}.x:X)^j}] : (x : X)^{*n}[\mathsf{S}^i_{(\Gamma.X:\mathsf{U}.x:X)^j}] \\
\text{(new rule)} \qquad \equiv\, & x_{2^n} \qquad : \underbrace{X_{2^n}[\mathsf{S}^i_{(\Gamma.X:\mathsf{U}.x:X)^j}]\,(x)_n[\mathsf{S}^i_{(\Gamma.X:\mathsf{U}.x:X)^j}]}_{\equiv X_{2^n}\,(x)_n \text{ by the previous new rule}}
\end{aligned}$$

If $A \equiv \Pi(x : A).B$:

$$\begin{aligned}
(\Gamma.f : \Pi(x : A).B)^n \vdash \quad & f_{2^n}[\mathsf{S}^i_{(\Gamma.f)^j}] \\
\text{(η-expansion)} \qquad & \equiv \lambda(x : A)^n[\mathsf{S}^i_{\Gamma^j}].f_{2^n}[\mathsf{S}^i_{(\Gamma.f)^j}]\,(x)^n[\mathsf{S}^i_{\Gamma^j}] \\
\text{(new rule)} \qquad & \equiv \lambda(x : A)^n[\mathsf{S}^i_{\Gamma^j}].f_{2^n}\,(x)^n[\mathsf{S}^i_{(\Gamma.x:A)^j}\,\mathsf{S}^i_{\Gamma^j}] \\
& : \qquad \underbrace{(f : \Pi(x : A).B)^{*n}[\mathsf{S}^i_{(\Gamma.f)^j}]}_{\equiv \Pi(x:A)^n[\mathsf{S}^i_{\Gamma^j}].(y:B)^{*n}[\mathsf{S}^i_{\Gamma^j},(x)^n,(f\,x)_n[\mathsf{S}^i_{(\Gamma.f)^j}]]}
\end{aligned}$$

Similarly here, f_{2^n} is looked up directly.

If $A \equiv \Sigma(x : A).B$:

$$\begin{aligned}
(\Gamma.w : \Sigma(x : A).B)^n \vdash \quad & w_{2^n}[\mathsf{S}^i_{(\Gamma.w)^j}] \\
\text{(η-expansion)} \qquad & \equiv (\pi_0\, w_{2^n}[\mathsf{S}^i_{(\Gamma.w)^j}], \pi_1\, w_{2^n}[\mathsf{S}^i_{(\Gamma.w)^j}]) \\
\text{(new rule)} \qquad & \equiv (x_{2^n}[\mathsf{S}^i_{(\Gamma.x:A)^j}(\mathsf{id}_\Gamma, (\pi_0\, w)^n)], y_{2^n}[\mathsf{S}^i_{(\Gamma.x:A.y:B)^j}(\mathsf{id}_\Gamma, (\pi_0\, w)^n, (\pi_1\, w)^n)]) \\
& : \qquad \underbrace{(w : \Sigma(x : A).B)^{*n}[\mathsf{S}^i_{(\Gamma.w)^j}]}_{\equiv \Sigma(r:(x:A)^{*n}[(\mathsf{id}_\Gamma,(\pi_0\, w)_n)\mathsf{S}^i_{(\Gamma.w)^j}]).(y:B)^{*n}[(\mathsf{id}_\Gamma,(\pi_0\, w)_n,r,(\pi_1\, w)_n)\mathsf{S}^i_{(\Gamma.w)^j}]}
\end{aligned}$$

Here we split the job of looking up w_{2^n} in the context into solving two smaller problems: looking up x_{2^n} and y_{2^n}.

For base types such $A \equiv \mathbb{N}$, $(x : A)^{*n}$ is symmetric, so we just add the rule $(x : A)^{*n}[\mathsf{S}] \equiv (x : A)^{*n}$ and $x_{2^n}[\mathsf{S}^i_{(\Gamma.x:\mathbb{N})^j}] \equiv x_{2^n}$.

1.4 Kan conditions

1.4.1 New definition of $\sim_{\mathsf{U}^*}$

So far $\sim_{A^*[\mathsf{R}]}$ is reflexive (by refl), however transitivity and symmetry cannot be proven. To fix this, we redefine the relation $\sim_{\mathsf{U}^*}$ as an iterated Σ-type (it is a closed type so it does not depend on the context):

$$
\begin{aligned}
\sim_{\mathsf{U}^*} \equiv \lambda\, A\, B\,.\, \Sigma - \sim - \quad & : A \to B \to \mathsf{U} \\
\mathsf{coe}^0 \quad & : A \to B \\
\mathsf{coh}^0 \quad & : \Pi(x : A).x \sim \mathsf{coe}^0\, x \\
\mathsf{uncoe}^0 \quad & : \Pi(x : A, y : B, p : x \sim y).\mathsf{coe}^0\, x \sim_{B^*[\mathsf{R}_{B:\mathsf{U}}]} y \\
\mathsf{uncoh}^0 \quad & : \Pi(x : A, y : B, p : x \sim y) \\
& \qquad .\mathsf{coh}^0\, x \sim_{\substack{(x \sim y)^*[\mathsf{R}_{A:\mathsf{U}.\,B:\mathsf{U}.\,\sim:A\to B\to\mathsf{U}.\,x:A},\\ y_0 \mapsto \mathsf{coe}^0\, x, y_1 \mapsto y, y_2 \mapsto \mathsf{uncoe}^0\, x\, y\, p]}} p \\
\mathsf{coe}^1 \quad & : B \to A \\
\mathsf{coh}^1 \quad & : \Pi(y : B).\mathsf{coe}^1\, y \sim y \\
\mathsf{uncoe}^1 \quad & : \Pi(x : A, y : B, p : x \sim y).\mathsf{coe}^1\, y \sim_{A^*[\mathsf{R}_{A:\mathsf{U}}]} x \\
\mathsf{uncoh}^1 \quad & : \Pi(x : A, y : B, p : x \sim y) \\
& \qquad .\mathsf{coh}^1\, y \sim_{\substack{(x \sim y)^*[\mathsf{R}_{A:\mathsf{U}.\,B:\mathsf{U}.\,\sim:A\to B\to\mathsf{U}.\,y:B},\\ x_0 \mapsto \mathsf{coe}^1\, y, x_1 \mapsto x, x_2 \mapsto \mathsf{uncoe}^1\, x\, y\, p]}} p \\
: \mathsf{U} \to \mathsf{U} \to \mathsf{U} \quad &
\end{aligned}
$$

An element of $A \sim_{\mathsf{U}^*} B$ is not only a binary relation between A and B but a relation $\sim$ equipped with functions back and forth, proofs that these functions respect the relation and uniqueness conditions. coe corresponds to the first level Kan extension operation, coh to the Kan filling operation.

This definition of equality on the universe is equivalent to the following definition (which is in turn equivalent to the standard definition of equivalence in the book [4], see exercise 4.2 there):

$$
\begin{aligned}
\Gamma \vdash \lambda A\, B\,.\, \Sigma\, (- \sim - : A \to B \to \mathsf{U}).& \\
\Pi(x : A).\mathsf{isContr}(\Sigma(y : B).x \sim y)& \\
\times\ \Pi(y : B).\mathsf{isContr}(\Sigma(x : A).x \sim y)& \\
: \mathsf{U} \to \mathsf{U} \to \mathsf{U}\qquad\qquad\qquad\qquad &
\end{aligned}
$$

The above definition can be drawn like this (the third square is the type of the first two squares):

$$
\begin{tikzcd}
x \arrow[r, "p"] & y \\
x \arrow[u, "\mathsf{refl}\, x"] \arrow[r, "\mathsf{coh}^0\, x"'] \arrow[ur, phantom, "\mathsf{uncoh}^0\, x\, y\, p"] & \mathsf{coe}^0\, x \arrow[u, "\mathsf{uncoe}^0\, x\, y\, p"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "p"] & y \\
\mathsf{coe}^1\, y \arrow[u, "\mathsf{uncoe}^1\, x\, y\, p"] \arrow[r, "\mathsf{coh}^1\, y"'] \arrow[ur, phantom, "\mathsf{uncoh}^1\, x\, y\, p"] & y \arrow[u, "\mathsf{refl}\, y"']
\end{tikzcd}
\ :\
\begin{tikzcd}
A \arrow[r, "\sim"] & B \\
A \arrow[u, "\sim_{\mathsf{refl}\, A}"] \arrow[r, "\sim"'] \arrow[ur, phantom, "\lambda(x)^{=}_{1}.\ \sim_{(\mathsf{refl} \sim (x)^{=}_{1})}"] & B \arrow[u, "\sim_{\mathsf{refl}\, B}"']
\end{tikzcd}
$$

The target type of uncoh^0 can be written as follows (we omit some types from the context but give the variable names):

$$\begin{aligned}
&A, B : \mathsf{U}.\ \sim : A \to B \to \mathsf{U}.\mathsf{coe}^0.\mathsf{coh}^0.\mathsf{uncoe}^0.x : A.y : B.p : x \sim y \\
&\quad \vdash\ \mathsf{coh}^0\, x \sim_{\substack{(x\sim y)^*[\mathsf{R}_{A:\mathsf{U}.B:\mathsf{U}.\sim:A\to B\to \mathsf{U}.x:A},\\ y_0\mapsto \mathsf{coe}^0\, x, y_1 \mapsto y, y_2 \mapsto \mathsf{uncoe}^0\, x\, y\, p]}} p \\
&\quad \equiv\ \sim_{(\mathsf{refl}\,(\sim)\, x\, x\, (\mathsf{refl}\, x)\, (\mathsf{coe}^0\, x)\, y\, (\mathsf{uncoe}^0\, x\, y\, p))} (\mathsf{coh}^0\, x)\, p : \mathsf{U}
\end{aligned}$$

For a small type $A : \mathsf{U}$ we will use the relation already defined in section 1.1.4 as $\sim_{A^*}$. For U we just use the relation above defined. For non-inhabitable terms the new definition of $\sim_{\mathsf{U}^*}$ does not make a difference, so for such a $t : A$ we just use the $t^* : t[0] \sim_{A^*} t[1]$ defined in section 1.1.5.

However, we need to provide the coe, coh, uncoe, uncoh operations for every type A, that is, we need the following rules to be validated for $i \in \{0, 1\}$:

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^= \vdash \mathsf{coe}^i_{A^*} : A[i_\Gamma] \to A[(1-i)_\Gamma]} \qquad \frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^= \vdash \mathsf{coh}^i_{A^*} : \Pi(x : A[i_\Gamma]).x \overset{i}{\sim}_{A^*} \mathsf{coe}^i_{A^*}\, x}$$

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^= \vdash \mathsf{uncoe}^i_{A^*} : \Pi(x : A)^=.\mathsf{coe}^i_{A^*}\, x_i \sim_{A[1-i]^*[\mathsf{R}_{\Gamma^=}]} x_{1-i}}$$

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^= \vdash \mathsf{uncoh}^i_{A^*} : \Pi(x : A)^=.\mathsf{coh}^i_{A^*}\, x_i \sim_{\substack{(x_0 \sim_{A^*} x_1)^*[\mathsf{R}_{\Gamma^=.x_i:A[i]}, \\ x_{(1-i)0} \mapsto \mathsf{coe}^i_{A^*}\, x_i, \\ x_{(1-i)1} \mapsto x_{1-i}, \\ x_{(1-i)2} \mapsto \mathsf{uncoe}^i_{A^*}\, x_0\, x_1\, x_2]}} x_2}$$

$$x \overset{0}{\sim}_e y \equiv x \sim_e y \qquad x \overset{1}{\sim}_e y \equiv y \sim_e x$$

We will define these for any type by induction on the type structure.

coe, coh, uncoe and uncoh are just projection functions for the Σ-type.

1.4.2 Kan for base types

For a base type B such as $\mathbb{N}$ or the universe U we define the Kan operations as follows:

$$\begin{aligned}
&\Gamma^= \vdash \mathsf{coe}^i_{B^*} \equiv \mathsf{id} : B \to B \\
&\Gamma^= \vdash \mathsf{coh}^i_{B^*} \equiv \mathsf{refl} : \Pi(b : B)\,.\, \underbrace{b \overset{i}{\sim}_{B^*} \mathsf{coe}^i_{B^*}\, b}_{\equiv b \sim_{B^*} b}
\end{aligned}$$

For the case $B \equiv \mathsf{U}$, $\mathsf{coh}^i_{\mathsf{U}^*}\, A \equiv \mathsf{refl}\, A \equiv A^*[\mathsf{R}]$ can be expanded using rule 9.

1.4.3 Kan for Σ

Definition for Σ types:

$$\begin{aligned}
\Gamma^= \vdash \mathsf{coe}^i_{(\Sigma(x:A).B)^*} &\equiv \lambda(a, b).(\mathsf{coe}^i_{A^*}\, a, \mathsf{coe}^i_{B^*}[x_2 \mapsto \mathsf{coh}^i_{A^*}\, a]\, b) \\
&\quad : (\Sigma(x : A).B)[i] \to (\Sigma(x : A).B)[1-i] \\
\Gamma^= \vdash \mathsf{coh}^i_{(\Sigma(x:A).B)^*} &\equiv \lambda(a, b).(\mathsf{coh}^i_{A^*}\, a, \mathsf{coh}^i_{B^*}[x_2 \mapsto \mathsf{coh}^i_{A^*}\, a]\, b) \\
&\quad : \Pi(w : (\Sigma(x : A).B)[i])\,.\, w \overset{i}{\sim}_{(\Sigma(x:A).B)^*} \mathsf{coe}^i_{(\Sigma(x:A).B)^*}\, w
\end{aligned}$$

In the substitution $[x_2 \mapsto \mathsf{coh}^i_{A^*}\, a]$ we left the dependencies x_i implicit, i.e. it stands for $[x_0 \mapsto a, x_1 \mapsto \mathsf{coe}^i_{A^*}\, a, x_2 \mapsto \mathsf{coh}^i_{A^*}\, a]$.

1.4.4 Kan for Π

The coerce operation for Π types:

$$\begin{aligned}\Gamma^{=} \vdash \mathsf{coe}^{i}_{(\Pi(x:A).B)^*} &\equiv \lambda f.\lambda x.\mathsf{coe}^{i}_{B^*}[x_2 \mapsto \mathsf{coh}^{1-i}_{A^*}\, x]\,(f\,(\mathsf{coe}^{1-i}_{A^*}\, x))\\ &: (\Pi(x:A).B)[i] \to (\Pi(x:A).B)[i]\end{aligned}$$

We left the other two components of the substitution $[x_2 \mapsto \mathsf{coh}^{1-i}_{A^*}\, x]$ implicit.

The coherence operation for Π needs to have the following type:

$$\begin{aligned}\Gamma^{=} \vdash \mathsf{coh}^{i}_{(\Pi(x:A).B)^*} :\ & \Pi\,(f : \Pi(x : A[i]).B[i])\\ &\quad .\, f \overset{i}{\sim}_{(\Pi(x:A).B)^*} \lambda x.\mathsf{coe}^{i}_{B^*}[x_2 \mapsto \mathsf{coh}^{1-i}_{A^*}\, x]\,(f\,(\mathsf{coe}^{1-i}_{A^*}\, x))\\ \equiv\ & \Pi\,(f : (\Pi(x:A).B)[i], x_0 : A[0], x_1 : A[1], x_2 : x_0 \sim_{A^*} x_1)\\ &\quad .\, f\, x_i \overset{i}{\sim}_{B^*} \mathsf{coe}^{i}_{B^*}[x_2 \mapsto \mathsf{coh}^{1-i}_{A^*}\, x_{1-i}]\,(f\,(\mathsf{coe}^{1-i}_{A^*}\, x_{1-i}))\end{aligned}$$

We explain in detail how to define $\mathsf{coh}^{0}_{(\Pi(x:A)B)^*}\, f\, x_0\, x_1\, x_2$, the other direction is symmetric. It is intuitive to think about a context $(A^{=})^{=}$ as a context of squares, and we present our construction using this intuition, see section 1.2.

We start in the context $\Gamma^{=}.f : (\Pi(x:A).B)[0].x_0 : A[0].x_1 : A[1].x_2 : x_0 \sim_A x_1$. First we define the following incomplete square, and fill it from top to bottom so that we get a term $r : x_0 \sim_{A^*[\mathsf{R}_\Gamma 0_\Gamma]} \mathsf{coe}^{1}_{A^*}\, x_1$.

$$\begin{tikzcd} x_1 \arrow[r, "\mathsf{refl}\, x_1"] & x_1 & & A[1] \arrow[r, "\sim_{\mathsf{refl}\, A[1]}"] & A[1] \\ x_0 \arrow[u, "x_2"] \arrow[r, dashed, "r"'] & \mathsf{coe}^{1}_{A^*}\, x_1 \arrow[u, "\mathsf{coh}^{1}_{A^*}\, x_1"'] & : & A[0] \arrow[u, "\sim_{A^*}"] \arrow[r, "\sim_{\mathsf{refl}\, A[0]}"'] & A[0] \arrow[u, "\sim_{A^*}"'] \end{tikzcd}$$

The two sides of the square are given by the following substitution:

$$\rho : \Gamma^{=}.f : (\Pi(x:A).B)[0].x_0 : A[0].x_1 : A[1].x_2 : x_0 \sim_{A^*} x_1 \to (\Gamma^{=}.x_0 : A[0].x_1 : A[1])^{=}$$
$$\begin{aligned}\rho \equiv (&(\mathsf{R}_\Gamma)^{=}, x_{00} \mapsto x_0, x_{01} \mapsto x_1, x_{20} \mapsto x_2,\\ &x_{10} \mapsto \mathsf{coe}^{1}_{A^*}\, x_1, x_{11} \mapsto x_1, x_{21} \mapsto \mathsf{coh}^{1}_{A^*}\, x_1)\end{aligned}$$

And r is computed by coerce for the type $x_0 \sim_{A^*} x_1$:

$$\Gamma^{=}.f : (\Pi(x:A).B)[0].x_0 : A[0].x_1 : A[1].x_2 : x_0 {\sim_{A^*}} x_1 \vdash r :\equiv \mathsf{coe}^{1}_{(x_0 \sim_{A^*} x_1)^*}[\rho]\,(\mathsf{refl}\, x_1)$$

Now we fill the following incomplete square from top to bottom which gives us the term that we need.

$$\begin{tikzcd} f\,(\mathsf{coe}^{1}_{A^*}\, x_1) \arrow[r, "\mathsf{coh}^{0}_{B^*}[x_2 \mapsto \mathsf{coh}^{1}_{A^*}\, x_1]\,(f\,(\mathsf{coe}^{1}_{A^*}\, x_1))"] & \mathsf{coe}^{0}_{B^*}[x_2 \mapsto \mathsf{coh}^{1}_{A^*}\, x_1]\,(f\,(\mathsf{coe}^{1}_{A^*}\, x_1)) \\ f\, x_0 \arrow[u, "f^*{[\mathsf{R}_{\Gamma^=}]}\, x_0\, (\mathsf{coe}^{1}_{A^*}\, x_1)\, r"] \arrow[r, dashed, "\mathsf{coh}^{0}_{(\Pi(x:A).B)^*}\, f\, x_0\, x_1\, x_2"'] & \mathsf{coe}^{0}_{B^*}[x_2 \mapsto \mathsf{coh}^{1}_{A^*}\, x_1]\,(f\,(\mathsf{coe}^{1}_{A^*}\, x_1)) \arrow[u, "\mathsf{refl}\,(\mathsf{coe}^{0}_{B^*}[x_2 \mapsto \mathsf{coh}^{1}_{A^*}\, x_1]\,(f\,(\mathsf{coe}^{1}_{A^*}\, x_1)))"'] \end{tikzcd}$$

We define the sides of the square by the substitution σ which has the following type:

$$\begin{aligned}\sigma : \Gamma^{=}.f : (\Pi(x : A).B)[\mathsf{0}].x_0 : A[\mathsf{0}].x_1 : A[\mathsf{1}].x_2 : x_0 \sim_{A^*} x_1 \\ \to (\Gamma.x : A)^{==}.(y_0 : B[\mathsf{0}])^{=}.(y_1 : B[\mathsf{1}])^{=}\end{aligned}$$

The part which gives $(x : A)^{==}$ is defined by the square that we just filled above, but swapped:

$$\begin{tikzcd}[column sep=huge]
\mathsf{coe}^1_{A^*}\, x_1 \arrow[r, "\mathsf{coh}^1_{A^*}\, x_1"] & x_1 \\
x_0 \arrow[u, "r \equiv \mathsf{coe}^1_{(x_0 \sim_{A^*} x_1)^*}[\rho]\,(\mathsf{refl}\, x_1)"] \arrow[r, "x_2"'] \arrow[ur, phantom, "(\mathsf{coh}^1_{(x_0 \sim_{A^*} x_1)^*}[\rho]\,(\mathsf{refl}\, x_1))[\mathsf{S}_{\Gamma.A}]"] & x_1 \arrow[u, "\mathsf{refl}\, x_1"']
\end{tikzcd}$$

The definition of σ:

$$\begin{aligned}\sigma \equiv (\mathsf{R}_{\Gamma^=}, & x_{00} \mapsto x_0, x_{01} \mapsto \mathsf{coe}^1_{A^*}\, x_1, x_{02} \mapsto r, \\ & x_{10} \mapsto x_1, x_{11} \mapsto x_1, x_{12} \mapsto \mathsf{refl}\, x_1 \\ & x_{20} \mapsto x_2, x_{21} \mapsto \mathsf{coh}^1_{A^*}\, x_1, x_{22} \mapsto (\mathsf{coh}^1_{(x_0 \sim_{A^*} x_1)^*}[\rho]\,(\mathsf{refl}\, x_1))[\mathsf{S}_{\Gamma.A}], \\ & y_{00} \mapsto f\, x_0, y_{01} \mapsto f\,(\mathsf{coe}^1_A\, x_1), y_{02} \mapsto f^*[\mathsf{R}_{\Gamma^=}]\, x_0\, (\mathsf{coe}^1_A\, x_1)\, r \\ & y_{10}, y_{11} \mapsto \mathsf{coe}^0_{B^*}[x_2 \mapsto \mathsf{coh}^1_{A^*}\, x]\,(f\,(\mathsf{coe}^1_{A^*}\, x)), \\ & y_{12} \mapsto \mathsf{refl}\,(\mathsf{coe}^0_{B^*}[x_2 \mapsto \mathsf{coh}^1_{A^*}\, x]\,(f\,(\mathsf{coe}^1_{A^*}\, x))))\end{aligned}$$

So the definition of coherence for Π is:

$$\Gamma^{=} \vdash \mathsf{coh}^0_{(\Pi(x:A).B)^*} \equiv \lambda f.\lambda x_0.\lambda x_1.\lambda x_2.\mathsf{coe}^1_{(y_0 \sim_{B^*} y_1)^*}[\sigma]\,(\mathsf{coh}^1_{A^*}\, x_1)$$

1.4.5 Swapping the universe

By redefining $A \sim_{\mathsf{U}^*} B$ to be the type of equivalences instead of relations, the presentation of $(X : \mathsf{U})^{*n}$ in section 1.2.2 changes, it is *not* true anymore that

$$(A : \mathsf{U})^n \vdash (A : \mathsf{U})^{*n} \equiv (x : A)_n \to \mathsf{U}.$$

However $(A : \mathsf{U})^{*n}$ still contains this information and we can project it out. For this we define a new operation specified by

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^n \vdash A^{\sim n} : (x : A)_n \to \mathsf{U}.}$$

Definition:

$$\begin{array}{lll} \Gamma^0 & \vdash A^{\sim 0} & \equiv A & : \mathsf{U} \\ \Gamma^{n+1} & \vdash A^{\sim n+1} & \equiv \lambda((x)_n)^{=}.\sim((A^{\sim n})^*\,((x)_n)^{=}) & : (x : A)_{n+1} \to \mathsf{U} \end{array}$$

Note that $\Gamma^{=} \vdash A^{\sim 1} \equiv \sim(A_2)$.

We unfold the type of A_{2^n} for the first few ns, that is, $(A : \mathsf{U})^{*n}$. We only show one of the two symmetric parts of the universe, and we mark the outputs with red color on the picture.

$$
\begin{aligned}
&(A : \mathsf{U})^0 \vdash A : (A : \mathsf{U})^{*0} \equiv \mathsf{U} \\
&(A : \mathsf{U})^1 \vdash A_2 : (A : \mathsf{U})^{*1} \equiv A_0 \sim_{\mathsf{U}^*} A_1 \\
&\equiv \Sigma \sim_{A_2} \quad : A_0 \to A_1 \to \mathsf{U} \\
&\qquad \mathsf{coe}_{A_2} \quad : A_0 \to A_1 \\
&\qquad \mathsf{coh}_{A_2} \quad : \Pi(x_0 : A_0).x_0 \sim_{A_2} \mathsf{coe}\, x_0 \\
&\qquad \mathsf{uncoe}_{A_2} : \Pi(x_0 : A_0.x_1 : A_0.x_2 : x_0 \sim_{A_2} x_1).\mathsf{coe}\, x_0 \sim_{\mathsf{r}\, A_1} x_1 \\
&\qquad \mathsf{uncoh}_{A_2} : \Pi(x_0 : A_0.x_1 : A_0.x_2 : x_0 \sim_{A_2} x_1) \\
&\qquad\qquad\qquad . \sim_{(\mathsf{r} \sim_{A_2} x_0\, x_0\, (\mathsf{r}\, x_0)\, (\mathsf{coe}_{A_2}\, x_0)\, x_1\, (\mathsf{uncoe}_{A_2}\, (x)^=))} (\mathsf{coh}_{A_2}\, x_0)\, x_2
\end{aligned}
$$

$$
\begin{tikzcd}
x_0 \arrow[r, "x_2"] & x_1 & & A_0 \arrow[r, "\sim_{A_2}"] & A_1 \\
x_0 \arrow[u, "\mathsf{r}\, x_0"] \arrow[r, "\mathsf{coh}_{A_2}\, x_0"'] \arrow[ur, phantom, "\mathsf{uncoh}_{A_2}\, (x)^="] & \mathsf{coe}_{A_2}\, x_0 \arrow[u, "\mathsf{uncoe}_{A_2}\, (x)^="'] & : & A_0 \arrow[u, "\sim_{\mathsf{r}\, A_0}"] \arrow[r, "\sim_{A_2}"'] \arrow[ur, phantom, "A^{\sim 2}[\mathsf{R}_{(A:\mathsf{U})^1}]"] & A_1 \arrow[u, "\sim_{\mathsf{r}\, A_1}"']
\end{tikzcd}
$$

Note that

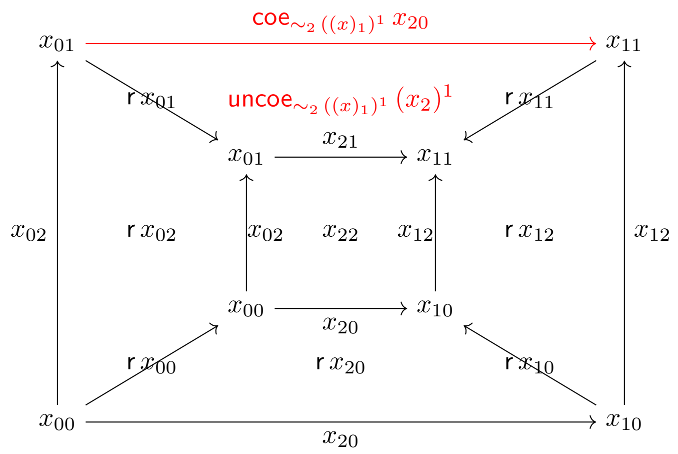
$$
\begin{aligned}
A^{\sim 2}[\mathsf{R}_{(A:\mathsf{U})^1}] &\equiv \lambda((x)_1)^1.\sim(\sim_2(A_{22}[\mathsf{R}_{(A:\mathsf{U})^1}])\, ((x)_1)^1) \\
&\equiv \lambda((x)_1)^1.\sim(\mathsf{r} \sim_{A_2} ((x)_1)^1).
\end{aligned}
$$

The next level:

$$
\begin{aligned}
&(A : \mathsf{U})^2 \vdash A_{22} : (A : \mathsf{U})^{*2} \equiv A_{20} \sim_{(A_0 \sim_{\mathsf{U}^*} A_1)^*} A_{21} \\
&\equiv \Sigma \sim_2(A_{22}) \quad : \Pi(x_0 : A_0 . x_1 : A_1)^{=} . (x_{00} \sim_{A_{20}} x_{10}) \sim_{\mathsf{U}^*} (x_{01} \sim_{A_{21}} x_{11}) \\
&\qquad \mathsf{coe}_2(A_{22}) \quad : \Pi(x_0 : A_0)^1 . \mathsf{coe}_{A_{20}}\, x_{00} \sim_{A_{12}} \mathsf{coe}_{A_{21}}\, x_{01} \\
&\qquad \mathsf{coh}_2(A_{22}) \quad : \Pi(x_0 : A_0)^1 . \mathsf{coh}_{A_{20}}\, x_{00} \sim_{(x_0 \sim_{A_2} \mathsf{coe}_{A_2}\, x_0)^*} \mathsf{coh}_{A_{21}}\, x_{01} \\
&\qquad \mathsf{uncoe}_2(A_{22}) : \Pi(x : A)^2 . \mathsf{uncoe}_{A_{20}}\, (x)^1[0] \sim_{(\mathsf{coe}_{A_2}\, x_0 \sim_{\mathsf{r}\, A_1} x_1)^*} \mathsf{uncoe}_{A_{21}}\, (x)^1[1] \\
&\qquad \mathsf{uncoh}_2(A_{22}) : \Pi(x : A)^2 . \sim_{(\sim_{(\mathsf{r} \sim_{A_2} x_0\, x_0\, (\mathsf{r}\, x_0)\, (\mathsf{coe}_{A_2}\, x_0)\, x_1\, (\mathsf{uncoe}_{A_2}\, (x)^1))}\, (\mathsf{coh}_{A_2}\, x_0)\, x_2)^*} \\
&\qquad\qquad\qquad\qquad (\mathsf{uncoh}_{A_{20}}\, (x)^1[0])\, (\mathsf{uncoh}_{A_{21}}\, (x)^1[1]) \\
&\equiv \Sigma \sim_2(A_{22}) \quad : \Pi(x_0 : A_0 . x_1 : A_1)^{=} \\
&\qquad\qquad . \mathsf{let}\ B \equiv (\sim_2(A_{22})\, (x_0)^{=}\, (x_1)^{=})\ \mathsf{in} \\
&\qquad\qquad\quad \Sigma \sim_B \quad : x_{00} \sim_{A_{20}} x_{10} \to x_{01} \sim_{A_{21}} x_{11} \to \mathsf{U} \\
&\qquad\qquad\qquad \mathsf{coe}_B \quad : x_{00} \sim_{A_{20}} x_{10} \to x_{01} \sim_{A_{21}} x_{11} \\
&\qquad\qquad\qquad \mathsf{coh}_B \quad : \Pi(x_{20} : x_{00} \sim_{A_{20}} x_{10}) . x_{20} \sim_B \mathsf{coe}_B\, x_{20} \\
&\qquad\qquad\qquad \mathsf{uncoe}_B : \Pi(x_2 : x_0 \sim_{A_2} x_1)^{=} . \mathsf{coe}_B\, x_{20} \sim_{\mathsf{r}\, (x_{01} \sim_{A_{21}} x_{11})} x_{21} \\
&\qquad\qquad\qquad \mathsf{uncoh}_B : \Pi(x_2 : x_0 \sim_{A_2} x_1)^{=} \\
&\qquad\qquad\qquad\qquad . \sim_{(\mathsf{r} \sim_B x_{20}\, x_{20}\, (\mathsf{r}\, x_{20})\, (\mathsf{coe}_B\, x_{20})\, x_{21}\, (\mathsf{uncoe}_B\, (x_2)^1))}\ (\mathsf{coh}_B\, x_{20})\, x_{22} \\
&\qquad \mathsf{coe}_2(A_{22}) \quad : \Pi(x_0 : A_0)^1 . \mathsf{coe}_{A_{20}}\, x_{00} \sim_{A_{12}} \mathsf{coe}_{A_{21}}\, x_{01} \\
&\qquad \mathsf{coh}_2(A_{22}) \quad : \Pi(x_0 : A_0)^1 . \sim_{(\sim_2(A_{22})\, (x_0)^{=}\, (\mathsf{coe}_{A_2}\, x_0)^{=})} (\mathsf{coh}_{A_{20}}\, x_{00})\, (\mathsf{coh}_{A_{21}}\, x_{01}) \\
&\qquad \mathsf{uncoe}_2(A_{22}) : \Pi(x : A)^2 . \sim_{(\sim_2(A_{122}[\mathsf{S}_{(A:\mathsf{U})} = \mathsf{R}_{(A:\mathsf{U})^2}])\, (\mathsf{coe}_{A_{20}}\, x_{00})\, (\mathsf{coe}_{A_{21}}\, x_{01})\, (\mathsf{coe}_2(A_{22})\, (x_0)^{=})\, x_{10}\, x_{11}\, x_{12})} \\
&\qquad\qquad\qquad\qquad (\mathsf{uncoe}_{A_{20}}\, (x)^1[0])\, (\mathsf{uncoe}_{A_{21}}\, (x)^1[1]) \\
&\qquad \mathsf{uncoh}_2(A_{22}) : \Pi(x : A)^2 . \sim\Big(\sim_2\big(\sim_{22}(A_{222}[\mathsf{S}_{(A:\mathsf{U})^1}\, \mathsf{R}_{(A:\mathsf{U})^2}]) \\
&\qquad\qquad\qquad\qquad\qquad x_{00}\, x_{01}\, x_{02} \\
&\qquad\qquad\qquad\qquad\qquad x_{00}\, x_{01}\, x_{02} \\
&\qquad\qquad\qquad\qquad\qquad (\mathsf{r}\, x_{00})\, (\mathsf{r}\, x_{01})\ \overbrace{(x_{022}[\mathsf{S}_{A_0:\mathsf{U}.x_0:A_0}\, \mathsf{R}_{(A_0:\mathsf{U}.x_0:A_0)^1}])}^{\equiv (\mathsf{r}\, x_0)^*} \\
&\qquad\qquad\qquad\qquad\qquad (\mathsf{coe}_{A_{20}}\, x_{00})\, (\mathsf{coe}_{A_{21}}\, x_{01})\, (\mathsf{coe}_2(A_{22})\, (x_0)^{=}) \\
&\qquad\qquad\qquad\qquad\qquad x_{10}\, x_{11}\, x_{12} \\
&\qquad\qquad\qquad\qquad\qquad (\mathsf{uncoe}_{A_{20}}\, (x)^{=}[0])\, (\mathsf{uncoe}_{A_{21}}\, (x)^{=}[1])\, (\mathsf{uncoe}_2(A_{22})\, (x)^2)\big) \\
&\qquad\qquad\qquad\qquad (\mathsf{coh}_{A_{20}}\, x_{00})\, (\mathsf{coh}_{A_{21}}\, x_{01})\, (\mathsf{coh}_2(A_{22})\, (x_0)^{=}) \\
&\qquad\qquad\qquad\qquad x_{20}\, x_{21}\, x_{22} \Big) \\
&\qquad\qquad\qquad (\mathsf{uncoh}_{A_{20}}\, (x)^1[0])\, (\mathsf{uncoh}_{A_{21}}\, (x)^1[1])
\end{aligned}
$$

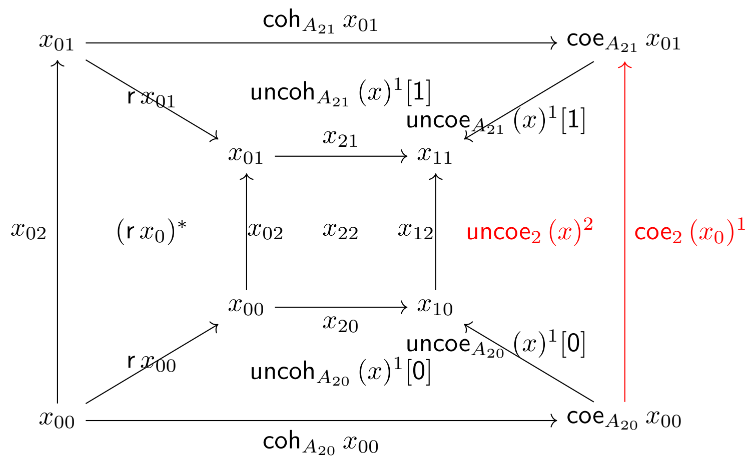
The cube described by $\mathsf{uncoh}_2(A_{22})$ is degenerate in it's z dimension (corresponding to the substitution $\mathsf{R}_{(A:\mathsf{U})^2}$), and its arguments are given in x, z, y order (which corresponds to swapping the dimensions y and z by the substitution $\mathsf{S}_{(A:\mathsf{U})^1}$).

The first two cubes are the contents of $(A : \mathsf{U})^{*2}$, the third is their type.



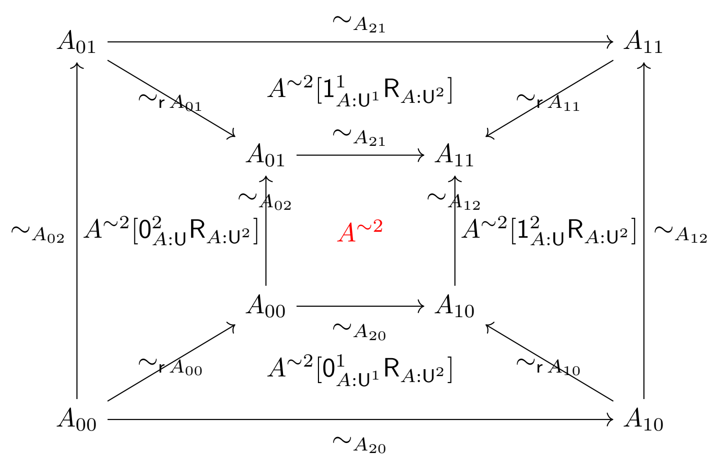
front face: $\mathsf{coh}_{\sim_2\,((x)_1)^1}\,x_{20}$

filler: $\mathsf{uncoh}_{\sim_2\,((x)_1)^1}\,(x_2)^1$



front face: $\mathsf{coh}_2\,(x_0)^1$

filler: $\mathsf{uncoh}_2\,(x)^2$



front face: $A^{\sim 2}$

filler: $A^{\sim 3}[\mathsf{R}_{(A:\mathsf{U})^2}]$

The type of the filler is $A^{\sim 3}[\mathsf{R}_{(A:\mathsf{U})^2}] \equiv \lambda((x)_2)^1.\sim_{(\mathsf{r}\,(\sim_2(A_{22}))\,((x)_2)^1)}$. The degenerate square types are:

$$
\begin{aligned}
A^{\sim 2}[\mathsf{0}^2_{A:\mathsf{U}}\mathsf{R}_{A:\mathsf{U}^2}] &\equiv \lambda((x)_1)^1.\sim_{(\mathsf{r}\sim_{A_{02}}((x)_1)^1)}\\
A^{\sim 2}[\mathsf{1}^2_{A:\mathsf{U}}\mathsf{R}_{A:\mathsf{U}^2}] &\equiv \lambda((x)_1)^1.\sim_{(\mathsf{r}\sim_{A_{12}}((x)_1)^1)}\\
A^{\sim 2}[\mathsf{0}^1_{A:\mathsf{U}^1}\mathsf{R}_{A:\mathsf{U}^2}] &\equiv \lambda((x)_1)^1.\sim_{(\mathsf{r}\sim_{A_{20}}((x)_1)^1)}\\
A^{\sim 2}[\mathsf{1}^1_{A:\mathsf{U}^1}\mathsf{R}_{A:\mathsf{U}^2}] &\equiv \lambda((x)_1)^1.\sim_{(\mathsf{r}\sim_{A_{21}}((x)_1)^1)}
\end{aligned}
$$

In the context $(A : \mathsf{U})^2$ we would like to define $A_{22}[\mathsf{S}_{A:\mathsf{U}}]$ in terms of A_{22}.

Its type is the following:

$$
\begin{array}{ll}
\multicolumn{2}{l}{(A : \mathsf{U})^2 \vdash A_{22}[\mathsf{S}_{A:\mathsf{U}}](A : \mathsf{U})^{*2}[\mathsf{S}_{A:\mathsf{U}}] \equiv A_{02} \sim_{(A_0 \sim_{\mathsf{U}^*} A_1)^*[\mathsf{S}_{A:\mathsf{U}}]} A_{12}} \\
\equiv \Sigma \sim_2(A_{22}[\mathsf{S}]) & : \Pi(x_0 : A)^=[\mathsf{0}_A].(x_1 : A)^=[\mathsf{1}_A] \\
& \quad .\mathsf{let}\ B \equiv (\sim_2(A_{22}[\mathsf{S}])\,(x_0)^=\,(x_1)^=)\ \mathsf{in} \\
& \quad\ \ \Sigma \sim_B \qquad : x_{00} \sim_{A_{02}} x_{10} \to x_{01} \sim_{A_{12}} x_{11} \to \mathsf{U} \\
& \quad\ \ \ \ \mathsf{coe}_B \qquad : x_{00} \sim_{A_{02}} x_{10} \to x_{01} \sim_{A_{12}} x_{11} \\
& \quad\ \ \ \ \mathsf{coh}_B \qquad : \Pi(x_{20} : x_{00} \sim_{A_{02}} x_{10}).x_{20} \sim_B \mathsf{coe}\, x_{20} \\
& \quad\ \ \ \ \mathsf{uncoe}_B \ : \Pi(x_{20} : x_{00} \sim_{A_{02}} x_{10}. \\
& \qquad\qquad\qquad\quad x_{21} : x_{01} \sim_{A_{12}} x_{11}. \\
& \qquad\qquad\qquad\quad x_{22} : x_{01} \sim_B x_{11}).\mathsf{coe}_B\, x_{20} \sim_{\mathsf{r}\,(x_{01} \sim_{A_{12}} x_{11})} x_{21} \\
& \quad\ \ \ \ \mathsf{uncoh}_B \ : \Pi(x_{20} : x_{00} \sim_{A_{02}} x_{10}. \\
& \qquad\qquad\qquad\quad x_{21} : x_{01} \sim_{A_{12}} x_{11}. \\
& \qquad\qquad\qquad\quad x_{22} : x_{01} \sim_B x_{11}) \\
& \qquad\qquad\qquad\quad .\sim_{(\mathsf{r} \sim_B x_{20}\, x_{20}\, (\mathsf{r}\, x_{20})\, (\mathsf{coe}_B\, x_{20})\, x_{21}\, (\mathsf{uncoe}_B\, (x_2)^1))} \\
& \qquad\qquad\qquad\quad (\mathsf{coh}_B\, x_{20})\, x_{22} \\
\quad \mathsf{coe}_2(A_{22}[\mathsf{S}]) & : \Pi(x_0 : A)^=[\mathsf{0}_A].\mathsf{coe}_{A_{02}}\, x_{00} \sim_{A_{21}} \mathsf{coe}_{A_{12}}\, x_{01} \\
\quad \mathsf{coh}_2(A_{22}[\mathsf{S}]) & : \Pi(x_0 : A)^=[\mathsf{0}_A].\sim_{(\sim_2(A_{22}[\mathsf{S}])\,(x_0)^=\,(\mathsf{coe}_{A_{02}}\, x_{00})\,(\mathsf{coe}_{A_{12}}\, x_{01})\,(\mathsf{coe}_2(A_{22}[\mathsf{S}])\,(x_0)^=))} \\
& \qquad\qquad\qquad (\mathsf{coh}_{A_{02}}\, x_{00})\,(\mathsf{coh}_{A_{12}}\, x_{01}) \\
\quad \mathsf{uncoe}_2(A_{22}[\mathsf{S}]) & : \Pi(x : A)^2[\mathsf{S}_A].\sim\big(\sim_2(A_{122}[\mathsf{S}_{(A:\mathsf{U})^=}\mathsf{S}^=_{A:\mathsf{U}}\mathsf{R}_{(A:\mathsf{U})^2}]) \\
& \qquad\qquad\qquad\quad (\mathsf{coe}_{A_{02}}\, x_{00})\,(\mathsf{coe}_{A_{12}}\, x_{01})\,(\mathsf{coe}_{2(A_{22}[\mathsf{S}])}\,(x_0)^=) \\
& \qquad\qquad\qquad\quad x_{10}\, x_{11}\, x_{12}\big) \\
& \qquad\qquad\qquad (\mathsf{uncoe}_{A_{02}}\,(x)^1[\mathsf{0}])\,(\mathsf{uncoe}_{A_{12}}\,(x)^1[\mathsf{1}]) \\
\quad \mathsf{uncoh}_2(A_{22}[\mathsf{S}]) & : \Pi(x : A)^2.\sim\Big(\sim_2\big(\sim_{22}(A_{222}[\mathsf{S}_{(A:\mathsf{U})^1}\mathsf{S}^1_{A:\mathsf{U}}\mathsf{R}_{(A:\mathsf{U})^2}]) \\
& \qquad\qquad\qquad\quad x_{00}\, x_{01}\, x_{02} \\
& \qquad\qquad\qquad\quad x_{00}\, x_{01}\, x_{02} \\
& \qquad\qquad\qquad\quad (\mathsf{r}\, x_{00})\,(\mathsf{r}\, x_{01})\,(x_{022}[\mathsf{S}_{A_0:\mathsf{U}.x_0:A_0}\mathsf{R}_{(A_0:\mathsf{U}.x_0:A_0)^1}\mathsf{S}_{A:\mathsf{U}}]) \\
& \qquad\qquad\qquad\quad (\mathsf{coe}_{A_{02}}\, x_{00})\,(\mathsf{coe}_{A_{12}}\, x_{01})\,(\mathsf{coe}_{A_{22}[\mathsf{S}]}\,(x_0)^=) \\
& \qquad\qquad\qquad\quad x_{10}\, x_{11}\, x_{12} \\
& \qquad\qquad\qquad\quad (\mathsf{uncoe}_{A_{02}}\,(x)^=[\mathsf{0}])\,(\mathsf{uncoe}_{A_{12}}\,(x)^=[\mathsf{1}])\,(\mathsf{uncoe}_{A_{22}[\mathsf{S}]}\,(x)^2)\big) \\
& \qquad\qquad\qquad (\mathsf{coh}_{A_{02}}\, x_{00})\,(\mathsf{coh}_{A_{12}}\, x_{01})\,(\mathsf{coh}_{A_{22}[\mathsf{S}]}\,(x_0)^=) \\
& \qquad\qquad\qquad x_{20}\, x_{21}\, x_{22}\Big) \\
& \qquad\qquad (\mathsf{uncoh}_{A_{02}}\,(x)^1[\mathsf{0}])\,(\mathsf{uncoh}_{A_{12}}\,(x)^1[\mathsf{1}])
\end{array}
$$

1.5 The identity type

We define $\Gamma \vdash a =_A b$ as $\Gamma \vdash a \sim_{A^*[\mathsf{R}_\Gamma]} b$.

We show that we have the non-dependent eliminator of the identity type usually called $\mathsf{transport}$ or subst i.e. the following rule is validated:

$$\frac{\Gamma \vdash P : A \to \mathsf{U} \quad \Gamma \vdash r : x \sim_{A^*[\mathsf{R}_\Gamma]} y \quad \Gamma \vdash u : P\,x}{\Gamma \vdash \mathsf{transport}_P\, r\, u : P\,y}$$

We have that P is a congruence:

$$\frac{\Gamma \vdash P : A \to \mathsf{U}}{\Gamma \vdash P^*[\mathsf{R}_\Gamma] : \Pi(x_0, x_1 : A, x_2 : x_0 \sim_{A^*[\mathsf{R}_\Gamma]} x_1).P\,x_0 \sim_{\mathsf{U}^*} P\,x_1}$$

And we define $\mathsf{transport}$ by using $P^*[\mathsf{R}]$:

$$\frac{\Gamma \vdash P : A \to \mathsf{U} \quad \Gamma \vdash r : x =_A y \quad \Gamma \vdash u : P\,x}{\Gamma \vdash \mathsf{transport}_P\, r\, u \equiv \mathsf{coe}^0_{P^*[\mathsf{R}_\Gamma]\,x\,y\,r}\, u : P\,y}$$

The computation rule of $\mathsf{transport}$ says that $\mathsf{transport}_P(\mathsf{refl}\,x) \equiv \mathsf{id}$. We have

$$\begin{aligned}
&\mathsf{transport}_P(\mathsf{refl}\,x)\\
\equiv\, &\mathsf{coe}^0_{P^*[\mathsf{R}_\Gamma]\,x\,x\,x^*[\mathsf{R}_\Gamma]}\\
\equiv\, &\mathsf{coe}^0_{(P\,x)^*[\mathsf{R}_\Gamma]}\\
\equiv\, &\mathsf{id}
\end{aligned}$$

The last step is justified by adding the following rule:

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma \vdash \mathsf{refl}\,A \equiv (\sim_{A^*[\mathsf{R}_\Gamma]}, \mathsf{id}, \mathsf{refl}, \mathsf{id}, \mathsf{refl}) : A \sim_{\mathsf{U}^*} A} \tag{9}$$

We also show that singletons are contractible i.e. we show how to construct the terms s and t of the following type:

$$\frac{\Gamma \vdash a, b : A \quad \Gamma \vdash r : a =_A b}{\begin{aligned}\Gamma \vdash\ (s,t) :\ & (a, \mathsf{refl}\,a) =_{\Sigma(x:A).a=_A x} (b, r)\\ \equiv\, & \Sigma(s : a \sim_{A^*[\mathsf{R}_\Gamma]} b).\mathsf{refl}\,a \sim_{(a \sim_{A^*[\mathsf{R}_\Gamma]} x)^*[\mathsf{R}_\Gamma, a, b, s]} r\end{aligned}}$$

s is constructed by filling the following incomplete square from bottom to top:

$$\begin{tikzcd}
a \arrow[r, dashed, "s"] & b \\
a \arrow[u, "\mathsf{refl}\,a"] \arrow[r, "\mathsf{refl}\,a"'] & a \arrow[u, "r"']
\end{tikzcd}$$

$$\begin{aligned}
&\rho :\equiv (\mathsf{R}_{\Gamma^=}\mathsf{R}_\Gamma, a, a, \mathsf{refl}\,a, a, b, r) : \Gamma \to (\Gamma^=.x_0 : A[0].x_1 : A[1])^=\\
&\Gamma \vdash s :\equiv \mathsf{coe}^0_{(x_0 \sim_{A^*} x_1)^*}[\rho](\mathsf{refl}\,a) : a \sim_{A^*[\mathsf{R}_\Gamma]} b
\end{aligned}$$

The coherence gives us the filler, however we need to swap it to get the type that we need:

$$\Gamma \vdash \mathsf{coh}^0_{(x_0 \sim_{A^*} x_1)^*}[\rho](\mathsf{refl}\,a) : \mathsf{refl}\,a \sim_{(x_0 \sim_{A^*} x_1)^*[\rho]} s$$

We define a substitution σ and compose it with $\mathsf{S}_{\Gamma.A}$ and we define t as the last term in the resulting substitution:

$$\Gamma \xrightarrow{\quad\sigma\quad} (\Gamma.A)^{==} \xrightarrow{\mathsf{S}_{\Gamma.A}} (\Gamma.A)^{==}$$

$$\begin{aligned}\sigma \equiv (&\mathsf{R}_{\Gamma^{=}}\mathsf{R}_{\Gamma}, a, a, \mathsf{refl}\, a,\\ &a, b, r,\\ &\mathsf{refl}\, a, \mathsf{coe}^{0}_{(x_0 \sim_{A^*} x_1)^*}[\rho](\mathsf{refl}\, a), \mathsf{coh}^{0}_{(x_0 \sim_{A^*} x_1)^*}[\rho](\mathsf{refl}\, a))\\ \Gamma \vdash t \equiv\ & x_{22}[\mathsf{S}_{\Gamma.A}\sigma] : \mathsf{refl}\, a \sim_{(a \sim_{A^*[\mathsf{R}_\Gamma]} x)^*[\mathsf{R}_\Gamma, a, b, s]} r\end{aligned}$$

2 Relation to the cubical set model

2.1 Higher Kan operations

For a type $\Gamma \vdash A : \mathsf{U}$ and a dimension n we are interested in the following two horns:

$$\Gamma^{n+1} \vdash (x : A)^{=}_{n}.(x : A)^{*n}[\mathsf{0}_{\Gamma^n.A_n}]$$
$$\Gamma^{n+1} \vdash (x : A)^{=}_{n}.(x : A)^{*n}[\mathsf{1}_{\Gamma^n.A_n}]$$

In the case of a 3-dimensional cube ($n+1 = 3$), the first one is the cube without the back face and the filling, the second is the cube without the front face and the filling (the excluded faces are x_{220} and x_{221}, respectively in figure 1).

Other horns, where the missing face is not one of the last two faces, but some other face, can be defined using the swap S.

A type is Kan if every horn can be filled. The higher Kan operations are given by the following terms:

$$\frac{\Gamma \vdash A : \mathsf{U} \quad \Gamma^{n+1} \vdash (\vec{a}, a) : (x : A)^{=}_{n}.(x : A)^{*n}[i]}{\Gamma^{n+1} \vdash \mathsf{coe}^{i}_{((x:A)^{*n})^*}\,(\vec{a}, a) : A^{*n}[1 - i][\vec{a}]}$$

$$\frac{\Gamma \vdash A : \mathsf{U} \quad \Gamma^{n+1} \vdash (\vec{a}, a) : (x : A)^{=}_{n}.(x : A)^{*n}[i]}{\Gamma^{n+1} \vdash \mathsf{coh}^{i}_{((x:A)^{*n})^*}\,(\vec{a}, a) : a \overset{i}{\sim}_{(x:A)^{*n}[\vec{a}]} \mathsf{coe}^{n+1}_{i}\,(\vec{a}, a)}$$

$$x \overset{0}{\sim}_e y \equiv x \sim_e y \qquad x \overset{1}{\sim}_e y \equiv y \sim_e x$$

Note that

$$\frac{\Gamma^{n}.(x : A)_n \vdash (x : A)^{*n} : \mathsf{U}}{\Gamma^{n+1}.(x : A)^{=}_{n} \vdash ((x : A)^{*n})^* : (x : A)^{*n}[\mathsf{0}_{\Gamma^n.A_n}] \sim_{\mathsf{U}^*} (x : A)^{*n}[\mathsf{1}_{\Gamma^n.A_n}].}$$

$((x : A)^{*n})^*$ is a tuple, the first element of which is a relation, the second is the coerce from $(x : A)^{*n}[\mathsf{0}]$ to $(x : A)^{*n}[\mathsf{1}]$, the third is the corresponding coherence operation etc.

For example, to compute the missing upper edge of the following horn:

$$\begin{tikzcd} x_{01} & x_{11} \\ x_{00} \arrow[u, "x_{02}"] \arrow[r, "x_{20}"'] & x_{10} \arrow[u, "x_{12}"'] \end{tikzcd}$$

we can use the coerce for the type $x_0 \sim_{A^*} x_1 \equiv (x : A)^{*1}$:

$$(\Gamma^=.x_0 : A[0].x_1 : A[1])^= \vdash \mathsf{coe}^0_{(x_0 \sim_{A^*} x_1)^*}\, x_{20} : x_{01} \sim_{A^*} [1]\, x_{11}$$

2.2 Internalisation of the cubical set model

Let $\mathcal{C}$ be the category of names and substitutions where we identify two objects if they have the same cardinality: the objects are natural numbers and $\mathcal{C}(m, n) \equiv \{f : \bar{m} \to \bar{n} + \bar{2} \mid \forall i, j : \bar{m}, k : \bar{n}. f\, i = \mathsf{inl}\, k = f\, j \to i = j\}$ where $\bar{n}$ is the set with n elements.

For each context Γ we have a functor $\mathcal{C} \to \mathbf{Sets}$, an object m is mapped to Γ^m and a morphism t is mapped to t_Γ. The morphisms are interpreted as follows. Above the line is the substitution in our syntax, below is the corresponding morphism in the base theory of the cubical set model of [3].

$$\frac{\mathsf{R}^{n-i}_{\Gamma^i} : \Gamma^n \to \Gamma^{n+1}}{(d_i) : \{d_0, ..., d_{i-1}, d_{i+1}, ..., d_n\} \to \{d_0, ..., d_n\}}$$

$\mathsf{R}^{n-i}_{\Gamma^i}$ adds a degenerate dimension d_i (in this dimension, the types of points are all the same and the types of the lines are homogeneous equalities). If $\Gamma^{n+1} \vdash w : A^{\sim n+1}\, \vec{a}$, then $\Gamma^n \vdash w[\mathsf{R}^{n-i}_{\Gamma^i}]$ is an $n+1$-dimensional cube just as w, but the i^{th} dimension is made degenerate. This doesn't mean that all the edges are reflexivity in that dimension, it only means that the types of lines are reflexivities.

$$\frac{\mathsf{0}^{n-i}_{\Gamma^i} : \Gamma^{n+1} \to \Gamma^n}{(d_i = 0) : \{d_0, ..., d_n\} \to \{d_0, ..., d_{i-1}, d_{i+1}, ..., d_n\}}$$

$\mathsf{0}^{n-i}_{\Gamma^i}$ projects the cube in dimension d_i to 0. Similarly, $\mathsf{1}^{n-i}_{\Gamma^i}$ projects the cube in dimension d_i to 1. If $x \in \Gamma$, then $x_{j_0 \cdots j_{n-1}} \mapsto x_{j_0 \cdots j_{i-1} 0 j_{i+1} \cdots j_{n-1}}$ by $\mathsf{0}^{n-i}_{\Gamma^i}$.

$$\frac{\mathsf{S}^{n-i}_{\Gamma^i} : \Gamma^{n+2} \to \Gamma^{n+2}}{(d_0 \mapsto d_0, ..., d_i \mapsto d_{i+1}, d_{i+1} \mapsto d_i, ..., d_{n+1} \mapsto d_{n+1}) : \{d_0, ..., d_{n+1}\} \to \{d_0, ..., d_{n+1}\}}$$

$\mathsf{S}^{n-i}_{\Gamma^i}$ swaps dimensions d_i and d_{i+1}.

In $\mathcal{C}$ all morphisms are compositions of face maps $\overline{m+1} \to \overline{m}$ and degeneracies (which include permutations). For each context Γ, we need to give a map $[\![-]\!]_\Gamma$ and a quote function q s.t. for all morphisms t, t' in $\mathcal{C}$, $t = \mathsf{q}[\![t]\!]$ and if $[\![t]\!] \equiv [\![t']\!]$, then $\mathsf{q}\,[\![t]\!] = \mathsf{q}\,[\![t']\!]$ (soundness and completeness).

A horn of dimension n for a closed type A is given by excluding elements from $(a : A)^n$. In the cubical set model [3] a horn is given by $J, x \subset I$. We don't have named dimensions, so in our case $I = \{0, ...n-1\}$. Now the horn given by J, x will be

$$\begin{aligned}\vec{a} = \{a_{i_0 \dots i_{n-1}} \mid\ & k \in J, x \\ & \text{if } k = x \text{ then } i_k = 0, \text{ otherwise } i_k \in \{0, 1\}, \\ & j \in \{0, ..., k-1, k+1, ..., n-1\}, \\ & i_j \in \{0, 1, 2\}\} \subset (a : A)^n.\end{aligned}$$

These are the variables in the context which are in the horn. The ($n-1$-dimensional) faces in this horn are:

$$\{a_{i_0 \ldots i_{k-1} 0 i_{k+1} \ldots i_{n-1}} \mid k \in J, x\} \cup \{a_{i_0 \ldots i_{k-1} 1 i_{k+1} \ldots i_{n-1}} \mid k \in J\}.$$

The type of a variable can be given generically. We use the overloaded notation $A_{i_0 \ldots i_{n-1}}$ for the higher relation which is part of $A_{i_0 \ldots i_{n-1}}$ in the context $(A : \mathsf{U})^n$. The arguments of $A_{i_0 \ldots i_{n-1}}$ are given as a set which should be interpreted as a lexicographically ordered list.

$$\begin{aligned} a_{i_0 \ldots i_{n-1}} : A^{\sim(\sum_{i_k=2}^{k\in\{0,\ldots,n-1\}} 1)} \Big\{ a_{j_0 \ldots j_{n-1}} \mid\ & l \in \{k \in \{0, ..., n-1\} \mid i_k = 2\}, \\ & j_l \in \{0,1\}, \\ & m \in \{0, ..., n-1\}, m \neq l, \\ & \text{if } i_m = 2 \text{ then } j_m \in \{0,1,2\}, \text{ otherwise } j_m = i_m \Big\}. \end{aligned}$$

3 Metatheoretic properties

In this section we prove some properties of the theory defined above.

Adding rule 6 is sound. This is shown by the diagrams below: the left diagram is the induction hypothesis, the middle one is the induction step, the right one is the base case.

$$\begin{tikzcd} \Delta^= \arrow[r, "b_\Delta"] \arrow[d, "\rho^="] & \Delta \arrow[d, "\rho"] \\ \Gamma^= \arrow[r, "b_\Gamma"] & \Gamma \end{tikzcd} \quad \begin{tikzcd} \Delta^= \arrow[r, "b_\Delta"] \arrow[d, "(\rho,u)^= \equiv (\rho^=, u[0], u[1], u^*)"] & \Delta \arrow[d, "(\rho,u)"] \\ \Gamma^=.x_0 : A[0].x_1 : A[1].x_0 \sim_{A^*} x_1 \arrow[r, "(b_\Gamma, x_b)"] & \Gamma.A \end{tikzcd} \quad \begin{tikzcd} \Delta^= \arrow[r, "b_\Delta"] \arrow[d, "()"] & \Delta \arrow[d, "()"] \\ \emptyset \arrow[r, "()"] & \emptyset \end{tikzcd}$$

5 holds trivially.

Proof of adding 8 being sound:

$$\begin{tikzcd} \Delta^= \arrow[d, "\rho^="] & \Delta \arrow[l, "\mathsf{R}_\Delta"'] \arrow[d, "\rho"] \\ \Gamma^= & \Gamma \arrow[l, "\mathsf{R}_\Gamma"] \end{tikzcd} \quad \begin{tikzcd} \Delta^= \arrow[d, "(\rho^=, u[0], u[1], u^*)"] & \Delta \arrow[l, "\mathsf{R}_\Delta"'] \arrow[d, "(\rho,u)"] \\ \Gamma^=.x_0 : A[0].x_1 : A[1].x_0 \sim_{A^*} x_1 & \Gamma.A \arrow[l, "(\mathsf{R}_\Gamma, x, x, \mathsf{refl}\ x)"] \end{tikzcd} \quad \begin{tikzcd} \Delta^= \arrow[d, "()"] & \Delta \arrow[l, "\mathsf{R}_\Delta"'] \arrow[d, "()"] \\ \emptyset & \emptyset \arrow[l, "()"] \end{tikzcd}$$

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