# Notes on Definability and Kripke Logical Relations

**February 11, 2000**

Thorsten Altenkirch
alti@informatik.uni-muenchen.de

Department of Computer Science, University of Munich

## 1 Introduction

The question how to characterize the definable elements in models of the typed $\lambda$-calculus was first answered by Plotkin and Statman [Plo73,Plo80,Sta82]. Plotkin showed that in full models with infinite base type elements invariant under logical relations are definable upto level 2 of the type hierarchy and that Kripke logical relations characterize definable elements at all types. It was left open whether this characterization can be extended to arbitrary Henkin models and how to characterize relative definability. The first question was answered by Jung and Tiuryn who showed that invariance under Kripke logical relations with varying arity characterize definability in arbitrary Henkin models [JT93]. Alimohamed extended this result to categorical models and also showed that Kripke logical relations with varying arity can be used to characterize relative definability [Ali95]. The result by Loader shows that Kripke logical relations with fixed arity are not sufficient to characterize definablity over full finite models [Loa]. Kripke logical relations with varying arity were also used by O'Hearn and Riecke to define a semantic version of the fully abstract model of PCF [HR95] building on results by Sieber [Sie92]. Recently, Fiore and Simpson showed that Grothendieck logical relations, which are a Kripke Logical Relations with varying arity with an additional locality condition derived from sheaf theory can be used to characterize definable elements in a typed $\lambda$-calculus with coproducts [FS99].

We present the completeness proof of [JT93] in a slightly modernized form, inspired by the presentation in [FS99].

## 2 $\lambda$-calculus and Henkin models

The set of types Ty is given by a base type $\mathbf{o}$ and arrow types $\sigma \to \tau$. A type context $\Gamma \in \mathrm{Con}$ is a finite sequence of assumptions of the form $x : \sigma$ where $x$ is a term variable and $\sigma \in \mathrm{Ty}$. To simplify reasoning we use the following abbrevations:

$$\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$$
$$\Delta = y_1 : \tau_1, \ldots, y_m : \tau_m$$

We assume the usual rules for the simply typed $\lambda$ calculus:

$$\frac{}{\Gamma, x : \sigma, \Delta \vdash x : \sigma}\text{var} \qquad \frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x^\sigma.t : \sigma \to \tau}\text{lam} \qquad \frac{\Gamma \vdash t : \sigma \to \tau \quad \Gamma \vdash u : \sigma}{\Gamma \vdash tu : \tau}\text{app}$$

We shall always identify terms upto $\beta\eta$-equality.

An applicative structure is a pair $(\llbracket - \rrbracket, \mathbf{app})$, s.t.

- $\llbracket \sigma \rrbracket$ is a set for $\sigma \in \mathrm{Ty}$,
- $\mathbf{app}_{\sigma\tau} \in \llbracket \sigma \to \tau \rrbracket \times \llbracket \sigma \rrbracket \to \llbracket \tau \rrbracket$.

We omit the annotations $\sigma, \tau \ldots$ if they are clear from the context.

The applicative structure is extensional if for $f, g \in \llbracket \sigma \to \tau \rrbracket$ we have that:

$$\frac{\forall x \in \llbracket \sigma \rrbracket.\mathbf{app}(f, x) = \mathbf{app}(g, x)}{f = g}$$

There are two (equivalent) ways to say that an extensional applicative structure is a Henkin model (or just $\lambda$-model): *combinatory models* and *environment models*.

### 2.1 Combinatory models

The structure is a typed combinatory algebra, i.e. there are constants

$$\begin{aligned} \mathbf{k}_{\sigma\tau} &\in \llbracket \sigma \to \tau \to \sigma \rrbracket \\ \mathbf{s}_{\sigma\tau\rho} &\in \llbracket (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho \rrbracket \end{aligned}$$

s.t.

$$\begin{aligned} \mathbf{app}(\mathbf{app}(\mathbf{k}, p), q) &= p \\ \mathbf{app}(\mathbf{app}(\mathbf{app}(\mathbf{s}, p), q), r) &= \mathbf{app}(\mathbf{app}(p, r), \mathbf{app}(q, r)) \end{aligned}$$

### 2.2 Environment models

We interpret contexts by environments:

$$\llbracket \Gamma \rrbracket = \Pi\{x_i \mid 1 \leq i \leq n\}.\llbracket \sigma_i \rrbracket$$

An environment models is given by an assignment $\llbracket - \rrbracket$ which assigns to every welltyped term $\Gamma \vdash t : \sigma$ a function

$$\llbracket t \rrbracket \in \llbracket \Gamma \rrbracket \to \llbracket \sigma \rrbracket$$

s.t.

$$\begin{aligned} \llbracket x \rrbracket(\rho) &= \rho(x) \\ \llbracket tu \rrbracket(\rho) &= \mathbf{app}(\llbracket t \rrbracket(\rho), \llbracket u \rrbracket(\rho)) \\ \mathbf{app}(\llbracket \lambda x^\sigma.t \rrbracket(\rho), p) &= \llbracket t \rrbracket(\rho \cup \{(x, p)\}) \qquad \text{where} \quad p \in \llbracket \sigma \rrbracket \end{aligned}$$

## 3 Kripke Logical Relations of varying arity

A Kripke Logical Relation (of varying arity) is given by a category $\mathbb{W}$ of worlds, a functor $\mathbf{a} : \mathbb{W} \to \mathbb{SET}$ (called the arity functor) and a family of relations:

$$R^{\mathbf{o}}(w) \subseteq \mathbf{a}(w) \to [\![\mathbf{o}]\!]$$

s.t.

$$\frac{f \in \mathbb{W}(v, w) \qquad h \in R^{\mathbf{o}}(w)}{h \circ \mathbf{a}(f) \in R^{\mathbf{o}}(v)}\ \text{mon}$$

We can extend the relation to a family of relations indexed over types $\sigma \in \mathrm{Ty}$

$$R^{\sigma}(w) \subseteq \mathbf{a}(w) \to [\![\sigma]\!]$$

It is sufficient to extend the relation over arrow types: given $h \in \mathbf{a}(w) \to [\![\sigma \to \tau]\!]$ we say $h \in R^{\sigma\to\tau}(w)$ iff

$$\frac{f \in \mathbb{W}(v, w) \qquad g \in R^{\sigma}(v)}{(k \in \mathbf{a}(v) \mapsto \mathbf{app}((h \circ \mathbf{a}(f))(k), g(k))) \in R^{\tau}(v)}$$

We also extend the relation to contexts $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n \in \mathrm{Con}$:

$$R^{\Gamma}(w) = \{h_1, \dots, h_n \mid \forall 1 \leq i \leq n . h_i \in R^{\sigma_i}(w)\}$$

**Lemma 1.** *The condition (mon) holds for all types:*

$$\frac{f \in \mathbb{W}(v, w) \qquad h \in R^{\sigma}(w)}{h \circ \mathbf{a}(f) \in R^{\sigma}(v)}$$

*Proof.* By induction over the structure of types. Given $f \in \mathbb{W}(v, w)$ and $h \in R^{\sigma\to\tau}(w)$. To show that $h \circ \mathbf{a}(f) \in R^{\sigma}(v)$ assume $f' \in \mathbb{W}(v', v)$ we have to show that

$$(k \in \mathbf{a}(v') \mapsto \mathbf{app}((h \circ \mathbf{a}(f) \circ \mathbf{a}(f'))(k), g(k))) \in R^{\tau}(v')$$

Since $\mathbf{a}(f) \circ \mathbf{a}(f') = \mathbf{a}(f \circ f')$ we only have to use the hypothesis with $f \circ f'$.

**Proposition 1 (Fundamental theorem of logical relations).** *Given $\Gamma \vdash t : \sigma$ and a Kripke logical relation $R$ we have*

$$\frac{w \in \mathbb{W} \quad \boldsymbol{h} \in R^{\Gamma}(w)}{(k \in \mathbf{a}(w) \mapsto [\![t]\!](x_i \mapsto h_i(k))) \in R^{\sigma}(w)}$$

*Proof.* We show the theorem by induction over the derivation of $\Gamma \vdash t : \sigma$:

**(var)** Given $\Gamma \vdash x_j : \sigma_j$: we have that

$$\begin{aligned} k \in \mathbf{a}(w) \mapsto [\![x_j]\!](x_i \mapsto h_i(k) &= k \in \mathbf{a}(w) \mapsto h_j(k) \\ &= h_j \\ &\in R^{\sigma_j}(w) \end{aligned}$$

by using the premise.

**(app)** Assume we have derived $\Gamma \vdash tu : \tau$ from $\Gamma \vdash t : \sigma \to \tau$ and $\Gamma \vdash u : \sigma$. From the ind.hyp. we know that

$$\begin{aligned} h &= k \in \mathbf{a}(w) \mapsto [\![t]\!](x_i \mapsto h_i(k)) \\ &\in R^{\sigma \to \tau}(w) \end{aligned}$$

We choose $f = 1^w \in \mathbb{W}(w, w)$ and

$$\begin{aligned} g &= k \in \mathbf{a}(w) \mapsto [\![u]\!](x_i \mapsto h_i(k)) \\ &\in R^{\sigma}(w) \end{aligned}$$

by the other ind.hyp. Now we know that

$$\begin{aligned} k \in \mathbf{a}(w) \mapsto \mathbf{app}((h \circ \mathbf{a}(1^w))(k), g(k)) \\ \in R^{\tau}(w) \end{aligned}$$

Since

$$\begin{aligned} \mathbf{app}((h \circ \mathbf{a}(1^w))(k), g(k)) &= \mathbf{app}(h(k), g(k)) \\ &= \mathbf{app}([\![t]\!](x_i \mapsto h_i(k)), [\![u]\!](x_i \mapsto h_i(k)) \\ &= [\![tu]\!](x_i \mapsto h_i(k)) \end{aligned}$$

we arrive at

$$k \in \mathbf{a}(w) \mapsto [\![tu]\!](x_i \mapsto h_i(k)) \in R^{\tau}(w)$$

**(lam)** Assume we have derived $\Gamma \vdash \lambda x^\sigma.t : \sigma \to \tau$ from $\Gamma.x : \sigma \vdash t : \tau$. Given the premises we have to show that

$$\begin{aligned} h &= (k \in \mathbf{a}(w) \mapsto [\![\lambda x^\sigma.t]\!](x_i \mapsto h_i(k))) \\ &\in R^{\sigma \to \tau}(w) \end{aligned}$$

we assume $f \in \mathbb{W}(v, w)$ and $g \in R^{\sigma}(v)$.

$$\begin{aligned} & j \in \mathbf{a}(v) \mapsto \mathbf{app}((h \circ \mathbf{a}(f))(j), g(j)) \\ &= j \in \mathbf{a}(v) \mapsto \mathbf{app}([\![\lambda x^\sigma.t]\!](x_i \mapsto (h_i \circ \mathbf{a}(f))(j)), g(j)) \\ &= j \in \mathbf{a}(v) \mapsto [\![t]\!](x_i \mapsto (h_i \circ \mathbf{a}(f))(j), x \mapsto g(j)) \end{aligned}$$

Using monotonicity we have that $h_i \circ \mathbf{a}(f) \in R^{\sigma_i}(v)$ and hence $(\boldsymbol{h}, g) \in R^{\Gamma, x:\sigma}(v)$. We apply the induction hypothesis to derive

$$j \in \mathbf{a}(v) \mapsto [\![t]\!](x_i \mapsto (h_i \circ \mathbf{a}(f))(j), x \mapsto g(j)) \in R^{\tau}(v)$$

**Corollary 1.** *Given $t : \sigma$, a Kripke logical relation $R$ and a world $w \in \mathbb{W}$*

$$(k \in \mathbf{a}(w) \mapsto [\![t]\!]) \in R^{\sigma}(w)$$

## 4 Completeness

In this section we shall construct a special Kripke logical relation which characterizes the definable elements. Let $\mathbb{TM}$ be the category with objects are contexts $\Gamma \in \mathrm{Con}$ and morphisms are substitutions.

$$\mathbb{TM}(\Gamma, \Delta) = \Pi\{y_i \mid 1 \leq i \leq m\}.\{t \mid \Gamma \vdash t : \sigma_i\}$$

upto $\beta\eta$-equality. Composition is composition of substitutions. $\mathbb{TM}$ is an initial CCC.

We use the following arity functor:

$$\begin{aligned}
\mathbf{a}(\Gamma) &= [\![\Gamma]\!] \\
\mathbf{a}(\boldsymbol{t} \in \mathbb{TM}(\Gamma, \Delta)) &\in [\![\Gamma]\!] \to [\![\Delta]\!] \\
&= \rho \in [\![\Gamma]\!] \mapsto \{x_i \mid 1 \leq i \leq m\} \mapsto [\![t_i]\!](\rho)
\end{aligned}$$

We define a Kripke logical relation by

$$R^{\mathbf{o}}(\Gamma) = \{\rho \in [\![\Gamma]\!] \mapsto [\![t]\!]\rho \mid \Gamma \vdash t : \mathbf{o}\}$$

We now verify that $R$ corresponds to definability:

**Lemma 2.**

$$\frac{h \in R^{\sigma}(\Gamma)}{\exists \Gamma \vdash t : \sigma \quad [\![t]\!] = h}\, q \qquad \frac{\Gamma \vdash t : \sigma}{[\![t]\!] \in R^{\sigma}(\Gamma)}\, \mathrm{u}$$

*Proof.* By induction over the structure of $\sigma$: the case $\mathbf{o}$ follows directly from the definition, hence we consider $\sigma \to \tau$:

**q** Given $h \in R^{\sigma \to \tau}(\Gamma)$, i.e.

$$\forall \boldsymbol{s} \in \mathbb{TM}(\Delta, \Gamma). \forall g \in R^{\sigma}(\Delta).(\rho \in [\![\Delta]\!] \mapsto \mathbf{app}(h([\![\boldsymbol{s}]\!](\rho)), g(\rho))) \in R^{\tau}(\Delta)$$

We choose a fresh $x$ and consider $\Delta = \Gamma.x : \sigma$. We assign to $\boldsymbol{s}$ the weakening substitution $\pi_1 = x_i \mapsto x_i \in \mathbb{TM}(\Delta, \Gamma)$. We have $\Gamma.x : \sigma \vdash x : \sigma$ and hence by (u) for $\sigma$: $[\![x]\!] \in R^{\sigma}(\Gamma.x : \sigma)$.

Consider

$$\begin{aligned}
f &= \rho \in [\![\Gamma.x : \sigma]\!] \mapsto \mathbf{app}(h([\![\pi_1]\!](\rho)), [\![x]\!](\rho)) \\
&\in R^{\tau}(\Gamma.x : \sigma)
\end{aligned}$$

by (q) for $\tau$ we know that there is a term $\Gamma.x : \sigma \vdash t : \tau$ s.t. $[\![t]\!] = f$. We construct $\Gamma \vdash \lambda x^{\sigma}.t : \sigma \to \tau$ and show that $[\![\lambda x^{\sigma}.t]\!] = h$. Assume $\rho \in [\![\Gamma]\!]$ and $p \in [\![\sigma]\!]$:

$$\begin{aligned}
\mathbf{app}([\![\lambda x^{\sigma}.t]\!](\rho), p) &= [\![t]\!](\rho \cup \{(x, p)\}) \\
&= f(\rho \cup \{(x, p)\}) \\
&= \mathbf{app}(h([\![\pi_1]\!](\rho \cup \{(x, p)\})), [\![x]\!](\rho \cup \{(x, p)\})) \\
&= \mathbf{app}(h(\rho), p)
\end{aligned}$$

by (ext) $h(\rho) = [\![\lambda x^{\sigma}.t]\!](\rho)$ and since $\rho$ was chosen arbitrary: $h = [\![\lambda x^{\sigma}.t]\!]$.

**u** Given $\Gamma \vdash t : \sigma \to \tau$ we want to show $[\![t]\!] \in R^{\sigma\to\tau}(\Gamma)$. Assume as given $\boldsymbol{s} \in \mathbb{TM}(\Delta, \Gamma)$ and $g \in R^{\sigma}(\Delta)$. By (q) for $\sigma$ we know that there is $\Delta \vdash u : \sigma$ s.t. $[\![u]\!] = g$. We are left to show that

$$\begin{aligned} f &= \rho \in [\![\Delta]\!] \mapsto \mathbf{app}([\![t]\!]([\![\boldsymbol{s}]\!](\rho)), g(\rho)) \\ &\in R^{\tau}(\Delta) \end{aligned}$$

We show that $f$ is definable:

$$\begin{aligned} \mathbf{app}([\![t]\!]([\![\boldsymbol{s}]\!](\rho)), g(\rho)) &= \mathbf{app}([\![t]\!]([\![\boldsymbol{s}]\!](\rho)), [\![u]\!](\rho)) \\ &= \mathbf{app}([\![t[\boldsymbol{s}]]\!](\rho)), [\![u]\!](\rho)) \\ &= [\![t[\boldsymbol{s}]u]\!](\rho) \end{aligned}$$

Hence $f = [\![t[\boldsymbol{s}]u]\!]$ with $\Delta \vdash t[\boldsymbol{s}]u : \tau$ and by (u) for $\tau$ we know that $f \in R^{\tau}(\Delta)$.

We get a general completeness result which is strong enough to characterize relative definability:

**Proposition 2.** *Given* $g \in [\![\Gamma]\!] \to [\![\sigma]\!]$ *s.t.*

$$\frac{\Delta \in \mathbb{TM} \quad \boldsymbol{h} \in R^{\Gamma}(\Delta)}{(\rho \in [\![\Delta]\!] \mapsto g(x_i \mapsto h_i(\rho))) \in R^{\sigma}(\Delta)}$$

*then there is a* $\Gamma \vdash t : \sigma$ *s.t.* $[\![t]\!] = g$.

*Proof.* We set$\Delta = \Gamma$ and use (u) to derive

$$\begin{aligned} \boldsymbol{h} &= [\![1^{\Gamma}]\!] \\ &\in R^{\Gamma}(\Gamma) \end{aligned}$$

Hence

$$g \in R^{\sigma}(\Gamma)$$

and using (q) we derive that there is a $\Gamma \vdash t : \sigma$ s.t. $[\![t]\!] = g$.

**Corollary 2.**

$$\frac{g \in R^{\sigma}() \qquad x \in \mathbf{a}()}{\exists t : \sigma \qquad [\![t]\!] = g(x)}$$

## References


[Ali95] Moez Alimohamed. A characterization of lambda definability in categorical models of implicit polymorphism. *Theoretical Computer Science*, 146:5–23, 1995.

[FS99] Marcelo Fiore and Alex Simpson. Lambda definability with sums via grothendieck logical relatio ns. In *Proceedings TLCA 99*, number 1581 in LNCS, pages 147–161. Springer, 1999.

[HR95] Peter W. Hearn and Jon G. Riecke. Kripke logical relations and pcf. *Information and Computation*, 120, 1995.

[JT93] A. Jung and J. Tiuryn. A new characterization of lambda definability. In M. Bezem and J. F. Groote, editors, *Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*. Springer Verlag, 1993.

[Loa] Ralph Loader. The undecidability of lambda definibility. To appear in the Church Festschrift.

[Plo73] G. D. Plotkin. Lambda-definability and logical relations. Technical report, 1973.

[Plo80] G. D. Plotkin. Lambda definability in the full type hierarchy. In J P Seldin and J R Hindley, editors, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*. Academic Press, 1980.

[Sie92] Kurt Sieber. Reasoning about sequential functions via logical relations. In M. P. Fourman, P. T. Johnstone, and A. M. Pitts, editors, *Proc. LMS Symposium on Applications of Categories in Computer Science, Durham 1991*, volume 177 of *LMS Lecture Note Series*, pages 258–269. Cambridge University Press, 1992. LMS Lecture Notes Series, 177.

[Sta82] R. Statman. Completeness, invariance and $\lambda$-definability. *J. Symbolic Logic*, 47(1):17–26, 1982.