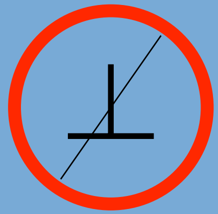
Stop thinking about bottoms when writing programs . . .



Thorsten Altenkirch
University of Nottingham

Trouble with $\perp$

$$(*) :: \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$0 \quad * \, n = 0$$
$$(m+1) * n = m * n + n$$

$$x * y = y * x \qquad ?$$

No, because

$$\begin{aligned} 0 * \perp &= 0 \\ \perp * 0 &= \perp \end{aligned}$$

Trouble with $\perp$ . . .

- Many useful algebraic properties do not hold.
- Correctness proofs get obliterated with reasoning about $\perp$.
- Do we actually care about non-terminating programs?
- Programs are **not** natural phenomena. . .
- Programs are **constructed**!

Do we need $\perp$ to be lazy?

$$\mathit{from} :: \mathbb{N} \to [\mathbb{N}]$$
$$\mathit{from}\ n = n : (\mathit{from}\ (n+1))$$

- *from* is total, **if** we interpret lists as a terminal coalgebra.

$$[A] = \nu X.1 + A \times X$$

data vs codata

$evenLength :: [a] \to \mathbf{Bool}$

$evenLength\ [\,] = \text{True}$

$evenLength\ (a : as) = \neg\,(evenLength\ n)$

- $evenLength$ is total, ...
- **if** we interpret lists as initial algebra:

$$[A] = \mu X.1 + A \times X$$

- Problem:

$$evenLength\ (from\ 0) = \bot$$

data vs codata

- Finite lists
  $\mathbf{data}\ [a] = [\,] \mid a : [a]$
- Potentially infinite lists:
  $\mathbf{codata}\ [a]^\omega\ a = [\,] \mid a : [a]^\omega$
- Better types
  $from :: \mathbb{N} \to [a]^\omega$
  $evenLength :: [a] \to \mathbf{Bool}$
- $evenLength\ (from\ 0)$ doesn't typecheck.

Can we always avoid $\bot$?

$\mathbf{data\ SK} = \text{S} \mid \text{K} \mid \mathbf{SK} :@\ \mathbf{SK}$

$nf :: \mathbf{SK} \to \mathbf{SK}$

$nf\ \text{S} = \text{S}$

$nf\ \text{K} = \text{K}$

$nf\ (t :@\ u) = (nf\ t)@(nf\ u)$

$(@) :: \mathbf{SK} \to \mathbf{SK} \to \mathbf{SK}$

$\text{K}\ @t = \text{K} :@\ t$

$(\text{K} :@\ t)\ @u = t$

$\text{S}\ @t = \text{S} :@\ t$

$(\text{S} :@\ t)\ @u = (\text{S} :@\ t) :@\ u$

$((\text{S} :@\ t) :@\ u)@v = (t@v)@(u@v)$

Computational Reals

- Define computational reals ($\mathbb{R}$) using Cauchy sequences.
- We cannot implement
  $pos :: \mathbb{R} \to \mathbf{Bool}$
- Indeed, all total computable functions of type $\mathbb{R} \to \mathbf{Bool}$ are constant (Brouwer).
- However, there are perfectly reasonable partial implementations of pos.

We need $\bot$ for:

- Interpreters.
- Functions on $\mathbb{R}$.
- more examples ?

Epigram

- Epigram is a dependently typed programming language...
- All Epigram programs are total (i.e. no $\bot$).
- It is **not** a programming language in Peter Mosses sense.
- because not all computable functions can be expressed.
- I am going to show how we can fix this. . .
- without making Epigram partial.

Monads. . .

- A monad $m :: * \to *$ is given by
  $$\begin{aligned} &\mathit{return} :: a \to m\ a \\ &(\geqslant) \quad :: (m\ a) \to (a \to m\ b) \to m\ b \end{aligned}$$
  subject to some equations.
- We can use monads to *encapsulate* effects (e.g. state)
  $$\begin{aligned} &\mathit{newIORef}\ :: a \to \mathbf{IO}\ (\mathbf{IORef}\ a) \\ &\mathit{readIORef} :: \mathbf{IORef}\ a \to \mathbf{IO}\ a \\ &\mathit{writeIORef} :: \mathbf{IORef}\ a \to a \to \mathbf{IO}\ () \end{aligned}$$
- and to *model* effects (e.g. state) :
  $$\begin{aligned} &\mathbf{data}\ \mathbf{ST}\ s\ a = \mathrm{M}\ (s \to (a, s)) \\ &\mathbf{instance}\ \mathit{Monad}\ (\mathbf{ST}\ s)\ \mathbf{where} \\ &\quad \mathit{return}\ a \qquad = \mathrm{M}\ (\lambda s \to (a, s)) \\ &\quad (\mathbf{ST}\ f) \gg\!= g = \mathrm{M}\ (\lambda s \to \mathbf{let}\ (a, s') = f\ s \\ &\qquad\qquad\qquad\qquad\qquad\quad \mathrm{M}\ g' = g\ a \\ &\qquad\qquad\qquad\qquad\qquad \mathbf{in}\ g'\ s') \end{aligned}$$

The Delay monad

$$\begin{aligned} &\mathbf{codata}\ \mathbf{D}\ a = \mathrm{Now}\ a \mid \mathrm{Later}\ (\mathbf{D}\ a) \\ &\mathbf{instance}\ \mathit{Monad}\ \mathbf{D}\ \mathbf{where} \\ &\quad \mathit{return} = \mathrm{Now} \\ &\quad \mathrm{Now}\ a \gg\!= k \ \ = k\ a \\ &\quad \mathrm{Later}\ d \gg\!= k = \mathrm{Later}\ (d \gg\!= k) \\ &\bot :: \mathbf{D}\ a \\ &\bot = \mathrm{Later}\ \bot \end{aligned}$$

Iteration with Delay

$$
\begin{aligned}
&rep :: (a \to \mathbf{D}\ (Either\ b\ a)) \to a \to \mathbf{D}\ b\\
&rep\ k\ a = k\ a \gg\!= \lambda ba \to\\
&\quad \mathbf{case}\ ba\ \mathbf{of}\\
&\qquad \mathrm{Left}\ b \to \mathrm{Now}\ b\\
&\qquad \mathrm{Right}\ a \to \mathrm{Later}\ (rep\ k\ a)
\end{aligned}
$$

Fixpoints with Delay

$$
\begin{aligned}
&rec :: ((a \to \mathbf{D}\ b) \to (a \to \mathbf{D}\ b)) \to a \to \mathbf{D}\ b\\
&rec\ \phi\ a = aux\ (\lambda\_ \to \bot)\\
&\quad \mathbf{where}\ aux :: (a \to \mathbf{D}\ b) \to \mathbf{D}\ b\\
&\qquad aux\ k = race\ (k\ a)\ (\mathrm{Later}\ (aux\ (\phi\ k)))\\
&race :: (\mathbf{D}\ a) \to (\mathbf{D}\ a) \to (\mathbf{D}\ a)
\end{aligned}
$$

$$
\begin{aligned}
race\ (\mathrm{Now}\ a)\ \_ \qquad\qquad &= \mathrm{Now}\ a\\
race\ (\mathrm{Later}\ \_)\ (\mathrm{Now}\ a) &= \mathrm{Now}\ a\\
race\ (\mathrm{Later}\ d)\ (\mathrm{Later}\ d') &= \mathrm{Later}\ (race\ d\ d')
\end{aligned}
$$

From Delay to Partial

- $\mathbf{D}$ is too intensional…
- We can observe how fast a function terminates.
- Hence $rec\ f \neq f\ (rec\ f)$
- We define
$$\mathbf{P}\ a = \mathbf{D}\ a / \simeq$$
where $\simeq \subseteq \mathbf{D}\ a \times \mathbf{D}\ a$ identifies values with different finite delay.
- We have to show that $\gg\!=, rep, rec$ preserve $\simeq$.
- We have $rec\ f \simeq f\ (rec\ f)$
if f is ω-continuous,
however all definable f are.

Defining $\simeq$

- $(\downarrow) \subseteq \mathbf{D}\ a \times a$ is defined inductively.
$$\frac{}{\mathrm{Now}\ a \downarrow a} \qquad \frac{d \downarrow a}{\mathrm{Later}\ d \downarrow a}$$
- $$
\begin{aligned}
\sqsubseteq \quad &\subseteq \quad \mathbf{D}\ a \times \mathbf{D}\ a\\
d \sqsubseteq d' \quad &= \quad \forall a. d \downarrow a \implies d' \downarrow a
\end{aligned}
$$
- $$
\begin{aligned}
\simeq \quad &\subseteq \quad \mathbf{D}\ a \times \mathbf{D}\ a\\
d \simeq d' \quad &= \quad d \sqsubseteq d' \wedge d' \sqsubseteq d
\end{aligned}
$$

Deja vu ?

- Constructive Domain Theory!
- $\mathbf{P}\ a = a_\perp$
- Note that constructively

$$a_\perp \neq a + \{\perp\}$$

  because we cannot observe non-termination.
- $\mathbf{P}\ a$ and hence $a \to \mathbf{P}\ b$ are ωCPOs.
- $\mathit{rec}\ f = \sqcup_{i \in \mathbb{N}} f^i \perp$ the code before constructs $\sqcup$ in $a \to \mathbf{P}\ b$.

Conclusions and further work

- Using the partiality monad we can encapsulate partial programs in a total language.
- *Partiality is an effect*
- We can reason about partial programs at compile time using the definition of $\mathbf{P}\ a$.
- and we can execute non-terminating programs at run-time.
- In future Epigram could support partiality without giving up the advantages of having a total language for most programs.
- Still to do: recursive datatypes by a constructive implementation of the standard domain-theoretic construction.

Thank you

- Thanks to Conor McBride & the Epigram Team
  (James Chapman, Peter Morris, Wouter Swierstra)
  see `www.e-pig.org` for more information on Epigram.
- Acknowledgements to Tarmo Uustalu and Venanzio Capretta
  for joint work on a partial paper...
- Looking for my papers?
  Type “Thorsten” into google...