Should Extensional Type Theory be considered harmful?

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Why Type Theory ?

- Accept the BHK explanation of constructive connectives.
- Proofs are programs, e.g.

$$\frac{p \in A \wedge (B \vee C)}{d\,p \in A \wedge B \vee A \wedge C}$$

$$\begin{aligned} d\,(a, \mathrm{inl}\,b) &\equiv \mathrm{inl}(a, b) \\ d\,(a, \mathrm{inr}\,c) &\equiv \mathrm{inr}(a, c) \end{aligned}$$

- We want to reason about programs/proofs...

$$\forall p, q \in A \wedge (B \vee C).d\,p = d\,q \implies p = q$$

Type Theory

- Propositions = Types, Proofs = Programs
- Connectives:

Σ	$\exists, \wedge, \times$
Π	$\forall, \implies, \rightarrow$
$0, 1, 2$	False, True, Bool
$a = b$	Equality types
W	Inductive types
$\mathbf{Type}_i$	Universes

- $A \vee B = A + B = \Sigma x : 2.\text{if } x \text{ then } A \text{ else } B$
  where $2 = \{0, 1\}$

The “axiom” of choice

$$\frac{f \in \Pi a \in A.\Sigma b \in B.R\,a\,b}{\mathrm{ac}\,f \in \Sigma g \in A \to B.\Pi a \in A.R\,a\,(g\,a)}$$

$$\mathrm{ac}\,f \quad = \quad (\pi_1 \circ f, \pi_2 \circ f)$$

where

$$\frac{p \in \Sigma a \in A.B\,a}{\begin{array}{l}\pi_1\,p \in A \\ \pi_2\,p \in B\,(\pi_1 p)\end{array}}$$

$$\begin{array}{lcl}\pi_1(a,b) & \equiv & a \\ \pi_2(a,b) & \equiv & b\end{array}$$

Which Type Theory?

- Impredicative (CoC) vs predicative
- Extensional (ETT) vs intensional (ITT)

Equality in ITT

Definitional equality $a \equiv b$

Propositional equality $p \in a = b$

$$\frac{m, n \in \mathrm{Nat}}{m + n \in \mathrm{Nat}}$$

$$\begin{aligned} 0 + n &\equiv n \\ (\mathrm{succ}\, m) + n &\equiv \mathrm{succ}(m + n) \end{aligned}$$

$$n + 0 \not\equiv n$$

$$\frac{n \in \mathrm{Nat}}{l\, n \in n + 0 = n}$$

Equality in ITT

- Conversion rule:

$$\frac{b \in B\,a \quad a \equiv a'}{b \in B\,a'}$$

- Equality introduction:

$$\frac{a \in A}{\mathrm{refl} \in a = a}$$

- Equality elimination:

$$\frac{b \in B\,a \quad p \in a = a'}{\mathrm{subst}_B\,p\,b \in B\,a'} \qquad \frac{p, q \in a = b}{\mathrm{unique}\,p\,q \in p = q}$$

$$\begin{aligned} \mathrm{subst}\,\mathrm{refl}\,b &\equiv b \\ \mathrm{unique}\,\mathrm{refl}\,\mathrm{refl} &\equiv \mathrm{refl} \end{aligned}$$

Equality reflection in ETT

$$\frac{p \in a = b}{a \equiv b}$$

Hence we have:

$$\frac{b \in B\,a \quad p \in a = a'}{b \in B\,a'}$$

The “axiom” of extensionality

$$\frac{f, g \in A \to B \quad p \in \Pi a \in A. f\, a = g\, a}{\mathrm{ext}\, p \in f = g}$$

- $\mathrm{ETT} \vdash \mathrm{ext}$
- $\mathrm{ITT} \not\vdash \mathrm{ext}$
- We cannot just add ext to ITT, because this leads to irreducible constants:

$$\mathrm{subst}_B\,(\mathrm{ext} \ldots)\, 3 \in \mathrm{Nat}$$

where $B\, f \equiv \mathrm{Nat}$

Setoids in ITT

- A setoid $(A, \sim_A)$ is a pair of a type A and an equivalence relation $\sim_A$.
- Given $(A, \sim_A), (B, \sim_B)$ we define $(A \to B, \sim_{A \to B})$ where

$$f \sim_{A \to B} g \equiv \Pi a \in A. f\ a \sim_B g\ a$$

- We have to show $\dfrac{b \in B\,a \quad p \in a \sim_A a'}{\mathrm{subst}^{\sim}_{B}\, p\, b \in B\, a'}$ for every B.
- Setoids can also be used to **approximate** quotient types.
- Using setoids can blow up the theory, e.g. formal category theory.

Why not ETT?

1. EAC (extensional axiom of choice) implies EM (excluded middle).
2. We cannot add CT, because AC+EXT+CT is inconsistent.

$$\mathrm{ct} \in \Pi f \in \mathrm{Nat} \to \mathrm{Nat}.\Sigma n \in \mathrm{Nat}.n \Vdash f$$

3. Type checking is undecidable.

1. EAC implies EM

Given $(A, \sim_A), (B, \sim_B)$

$$\frac{R \in A \to B \to \mathbf{Prop}}{\mathrm{ExtR}\, R \equiv \Pi a, a' \in A, b, b' \in B. a \sim_A a' \to b \sim_B b' \to R\, a\, b \to R\, a'\, b'}$$

$$\frac{f \in A \to B}{\mathrm{ExtF}\, f \equiv \Pi a, a' \in A. a \sim_A a' \to f\, a \sim_B f\, a'}$$

$$\frac{f \in \Pi a \in A. \Sigma b \in B. \mathrm{ExtR}\, R \times R\, a\, b}{\mathrm{eac}\, f \in \Sigma g \in A \to B. (\mathrm{ExtF}\, g) \times \Pi a \in A. R\, a\, (g\, a)}$$

- Why is EAC consistent?
- EAC is not derivable in ETT!
- In ETT with quotient types we can apply AC to $A/\sim_A$ and $B/\sim_B$ but any $f \in \Pi a \in A/\sim_A . \Sigma b \in B/\sim_B . R\, a\, b$ will respect the equivalences.

2. AC+EXT+CT is inconsistent.

- $$\mathrm{ct} \in \Pi f \in \mathrm{Nat} \to \mathrm{Nat}.\Sigma n \in \mathrm{Nat}.n \Vdash f$$

  says: *We know how to compute every function.*
- Bracket types (Awodey,Bauer):

  $$\frac{p, q \in [A]}{p \equiv q}$$

  i.e. $[A]$ is propositional.
- $$\mathrm{wct} \in \Pi f \in \mathrm{Nat} \to \mathrm{Nat}.[\Sigma n \in \mathrm{Nat}.n \Vdash f]$$

  says: *Every function is computable.*
- Conjecture: WCT is consistent with ETT and doesn't imply any taboos.

Bracket types

Prop: type with at most one inhabitant.

$$\frac{A \in \mathbf{Type} \quad \Pi a, b \in A.a = b}{A \in \mathbf{Prop}}$$

$[-]$ is monadic:

$$\frac{a \in A}{\mathrm{return}\, a \in [A]} \qquad \frac{f \in A \to [B] \quad a \in [A]}{\mathrm{bind}\, f\ a \in [B]}$$

$[-]$ is invariant on Prop:

$$\frac{A \in \mathbf{Prop} \quad a \in [A]}{\mathrm{unbox}\, a \in A}$$

3. Type checking is undecidable.

In recent joint work with Conor McBride we have introduced *Observational Type Theory* (OTT).

- All computations terminate and definitional equality and type checking are decidable.
- Propositional equality is extensional, i.e. two objects are equal, if all observations about them agree.
- Propositional equality is substitutive.
- Canonicity holds: any closed term is definitionally reducible to a canonical value.
- OTT is currently being implemented as the core of the *Epigram 2* system

Justification for OTT

I suggest to differentiate between:

data Types which are defined by their constructors:

- Finite types
- Inductive types, e.g. Nat, W-types
- Σ-types

codata Types which are defined by their eliminators:

- Π-types
- Coinductive types, e.g. Streams, M-types
- Σ-types
- Quotient types

Contracts

Data comes with a *producer contract*: the producer promises that the data has only produced using the constructors.
Codata comes with a *consumer contract*: the consumer promises only to inspect the codata using the eliminators.
In Weak Type Theory both contracts can be made explicit:

Typoids for data A type + a predicate which expresses that the data is inductively generated with the constructors.

Setoids for codata A type + an equivalence relation which expresses the observational equivalence generated by the eliminators.

ITT is skewed; it is strong for data, but weak for codata.

Conclusions

- OTT is a decidable variant of Extensional Type Theory.
- There are no foundational problems with OTT (We hope).
- There are problems with ITT:
  - Pragmatically: it is awkward to formalize Mathematics in it.
  - Foundationally: it doesn't deal with codata properly.