A syntactical approach to weak ω-groupoids

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What are weak ω-groupoids ?

1st answer (excutive summary)

A higher dimensional generalisation of equivalence relations.

2nd answer

Read the paper!

3rd answer

Download the Agda code!

Why are we interested in weak ω-groupoids ?

- Vladimir Voevodsky proposed *Univalent Type Theory*.
- A refinement of Martin-Löf Type Theory ...
- ...where equality of types is isomorphism.
- (or more precisely: weak equivalence).
- Inspired by the homotopy interpretation of Type Theory.
- Enables new ways of abstract reasoning.
- Structures can become 1st class objects.

ω-groupoid model of Type Theory ?

- Weak ω-groupoids provide a key tool to study the metatheory of Univalent Type Theory.
- We are interested in a computational interpretation of the univalence principle.
- This could be achieved if we can provide an interpretation of Type Theory using weak ω-groupoids.
- This would generalize the elimination of extensionality using setoids (LICS99).

But what is . . . ?

- To develop such a model . . .
- we need a precise definition of weak ω-groupoids.
- Formalized in Type Theory.

Equality types

- Equality types are an example of weak ω-groupoids.
- Given A : Set and $a, b : A$ we can form a new set $a = b$: Set
- For any $a : A$ we have a canonical proof id : $a = a$.
- Using the eliminator J we can show that $=$ is an equivalence relation:

$$
\begin{aligned}
p^{-1} : b = a & \qquad (p : a = b) \\
p \circ q : a = c & \qquad (p : b = c, q : a = b)
\end{aligned}
$$

- Given equality proofs $p, q : a = b$ we can form a new type $p = q$: Set.
- Which equalities between equality proofs are provable?

Groupoids

- We cannot prove $p = q$ for $p, q : a = b$ (Uniqueness of Identity proofs) using only J.
- we can show that $=$ has the structure of **groupoid**:

$$\begin{aligned} \lambda &: \mathrm{id} \circ p = p \\ \rho &: p \circ \mathrm{id} = p \\ \alpha &: p \circ (q \circ r) = (p \circ q) \circ r \\ \kappa &: p^{-1} \circ p = \mathrm{id} \\ \kappa' &: p \circ p^{-1} = \mathrm{id} \end{aligned}$$

- It is a weak groupoid because the equalities do not hold strictly (definitionally) ...
- ...but only propositionally (given by proofs).

Higher dimensions

- Since we can iterate equality types we get an infinite tower of weak groupoids.
- However, we get many additional equalities.
- $\circ$ is functorial, we also have

$$\alpha \cdot \beta : p \circ q = p' \circ q' \qquad (\alpha : p = p', \beta : q = q')$$

- Satisfying the functor laws,

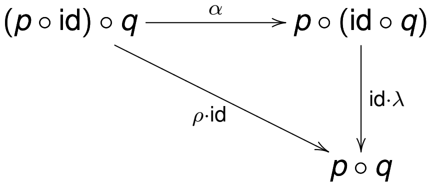
$$\mathsf{id} \cdot \mathsf{id} = \mathsf{id}$$
$$(\beta \circ \alpha) \cdot (\beta' \circ \alpha') = (\beta \cdot \beta') \circ (\alpha \cdot \alpha')$$

Coherence laws

- Another source of provable equalities are *coherence laws*
- There are two ways to show

$$(p \circ \mathrm{id}) \circ q = p \circ q$$

namely



which can be shown to be equal.

- In dimension 2 all coherence laws can be generated from 5 diagrams.
- In higher dimension it gets much more complicated . . .

Commutativity in higher dimensions (Eckmann-Hilton)

- Using the 2nd functor law we can also prove a form of commutativity:

$$\mathrm{comm}\, p\, q : p \circ q = q \circ p \qquad (p, q : \mathrm{id} = \mathrm{id})$$

$$\begin{array}{|c|c|}\hline p & q \\ \hline\end{array} = \begin{array}{|c|c|}\hline \mathrm{id} & q \\ \hline p & \mathrm{id} \\ \hline\end{array} = \begin{array}{|c|}\hline q \\ \hline p \\ \hline\end{array} = \begin{array}{|c|c|}\hline q & \mathrm{id} \\ \hline \mathrm{id} & p \\ \hline\end{array} = \begin{array}{|c|c|}\hline q & p \\ \hline\end{array}$$

- However, not all coherences are provable - we cannot derive

$$\mathrm{comm}\, p\, q \circ \mathrm{comm}\, q\, p = \mathrm{id} : q \circ p = q \circ p$$

From Equality to weak ω-groupoids

- What are the abstract properties of an equality?
- If we have uniqueness of identity proofs (UIP) this is just the notion of an equivalence relation.
- However, in the absence of UIP we need to make precise the notion of an ω-groupoid.
- There are a number of categorical definitions, due to Leinster, Penon and Batanin.
- However, they rely on the notion of strict ω-groupoid which is problematic in Type Theory.
- Here we propose an alternative characterisation in Type Theory.

Globular sets

We define a *globular set* G : Glob coinductively:

$$\begin{aligned} \mathsf{obj}_G &: \mathsf{Set} \\ \mathsf{hom}_G &: \mathsf{obj}_G \to \mathsf{obj}_G \to \infty\,\mathsf{Glob} \end{aligned}$$

Given globular sets A, B a morphism f : Glob(A, B) between them is given by

$$\begin{aligned} \mathrm{obj}_f^{\to} &: \mathsf{obj}_A \to \mathsf{obj}_B \\ \mathrm{hom}_f^{\to} &: \Pi a, b : \mathsf{obj}_A. \\ &\qquad \mathsf{Glob}(\mathsf{hom}_A\ a\, b, \mathsf{hom}_B(\mathrm{obj}_f^{\to}\ a, \mathrm{obj}_f^{\to}\ b)) \end{aligned}$$

As an example we can define the terminal object in $\mathbf{1}_{\mathsf{Glob}}$: Glob by the equations

$$\begin{aligned} \mathsf{obj}_{\mathbf{1}_{\mathsf{Glob}}} &= \mathbf{1}_{\mathsf{Set}} \\ \mathsf{hom}_{\mathbf{1}_{\mathsf{Glob}}}\ x\, y &= \mathbf{1}_{\mathsf{Glob}} \end{aligned}$$

The Identity Globular set

More interestingly, the globular set of identity proofs over a given set A, $\mathrm{Id}^{\omega}\,A : \mathsf{Glob}$ can be defined as follows:

$$\begin{aligned} \mathsf{obj}_{\mathrm{Id}^{\omega}\,A} &= A \\ \mathsf{hom}_{\mathrm{Id}^{\omega}\,A}\,a\,b &= \mathrm{Id}^{\omega}\,(a = b) \end{aligned}$$

Globular sets as a presheaf

Our definition of globular sets is equivalent to the usual one as a presheaf category over the diagram:

$$0 \underset{t_0}{\overset{s_0}{\rightrightarrows}} 1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} 2 \ldots n \underset{t_n}{\overset{s_n}{\rightrightarrows}} (n+1) \ldots$$

with the globular identities:

$$\begin{aligned} t_{i+1} \circ s_i &= s_{i+1} \circ s_i \\ t_{i+1} \circ t_i &= s_{i+1} \circ t_i \end{aligned}$$

A syntactic approach

- When is a globular set a weak ω-groupoid?
- We define a syntax for objects in a weak ω-groupoid.
- A globular set is a weak ω-groupoid, if we can interpret the syntax.
- This is reminiscient of environment λ-models.

The syntactical framework

Contexts

$$\mathsf{Con} : \mathsf{Set}$$

$$\frac{}{\varepsilon : \mathsf{Con}} \qquad \frac{C : \mathsf{Cat}\ \Gamma}{(\Gamma, C) : \mathsf{Con}}$$

Categories

$$\frac{\Gamma : \mathsf{Con}}{\mathsf{Cat}\ \Gamma : \mathsf{Set}}$$

$$\frac{}{\bullet : \mathsf{Cat}\ \Gamma} \qquad \frac{C : \mathsf{Cat}\ \Gamma \quad a, b : \mathsf{Obj}\ C}{C[\,a\,,\,b\,] : \mathsf{Cat}\ \Gamma}$$

Objects

$$\frac{C : \mathsf{Cat}\ \Gamma}{\mathsf{Obj}\ C, \mathsf{Var}\ C : \mathsf{Set}}$$

Interpretation

1. An assignment of sets to contexts:

$$\frac{\Gamma : \mathsf{Con}}{[\![\Gamma]\!] : \mathsf{Set}}$$

2. An assignment of globular sets to category expressions:

$$\frac{C : \mathsf{Cat}\ \Gamma \qquad \gamma : [\![\Gamma]\!]}{[\![C]\!]\ \gamma : \mathsf{Glob}}$$

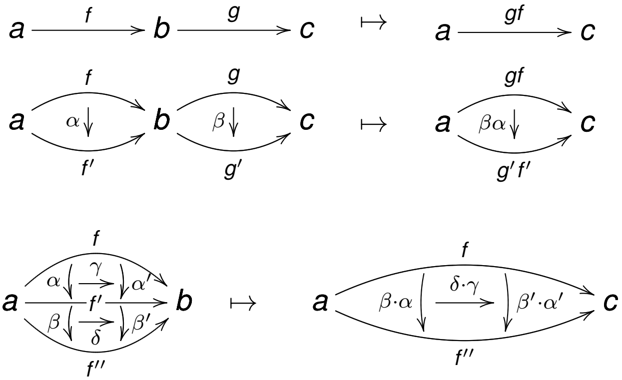
3. Assignments of elements of object sets to object expressions and variables

$$\frac{C : \mathsf{Cat}\ \Gamma \qquad A : \mathsf{Obj}\ C \qquad \gamma : [\![\Gamma]\!]}{[\![A]\!]\ \gamma : \mathsf{obj}_{[\![C]\!]\ \gamma}}$$

subject to some (obvious) conditions such as:

$$\begin{aligned} [\![\bullet]\!]\ \gamma &= G \\ [\![C[a,b]]\!]\ \gamma &= \mathsf{hom}_{[\![C]\!]\gamma}\ ([\![a]\!]\ \gamma)\ ([\![b]\!]\ \gamma) \end{aligned}$$

Composition



Telescopes

A telescope t : Tel C n is a path of length n from a category C of to one of its (indirect) hom-categories:

$$\frac{C : \mathrm{Cat}\ \Gamma \qquad n : \mathbb{N}}{\mathrm{Tel}\ C\ n : \mathrm{Set}}$$

We can turn telescopes into categories:

$$\frac{t : \mathrm{Tel}\ C\ n}{C +\!\!+\, t\ :\ \mathrm{Cat}\ \Gamma}$$

Formalizing composition

$$\frac{\alpha : \mathrm{Obj}(t \Downarrow) \qquad \beta : \mathrm{Obj}(u \Downarrow)}{\beta \circ \alpha : \mathrm{Obj}(u \circ t \Downarrow)}$$

is a new constructor of Obj where

$$\frac{t : \mathrm{Tel}\ (C[a,b])\ n \qquad u : \mathrm{Tel}\ (C[b,c])\ n}{u \circ t : \mathrm{Tel}\ (C[a,c])}$$

is a function on telescopes defined by cases

$$\bullet \circ \bullet C = \bullet \qquad u[a', b'] \circ t[a,b] = (u \circ t)[a' \circ a, b' \circ b]$$

Laws

For example the left unit law in dimension 1:

$$\mathrm{id}_b \circ f \quad = \quad f \ , \tag{1}$$

and in dimension 2.

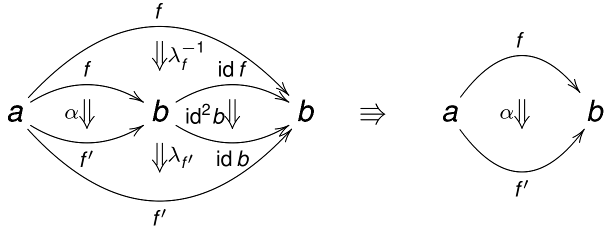
$$\mathrm{id}^2_b \circ \alpha \quad = \quad \alpha \ ,$$

where $\mathrm{id}^2_b = \mathrm{id}_{\mathrm{id}_b}$

In the strict case the 2nd equation only type-checks due to the first.

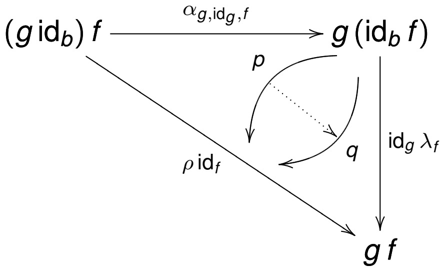
In the weak case we have to apply the previous isomorphism explicitely.

λ_α :



Coherence

Example:



In summary and full generality:

> *For any pair of coherence cells with the same domain and target, there must be a mediating coherence cell.*

Formalizing coherence

$$\frac{x : \text{Obj}\ C}{\text{hollow}\ x \ :\ \text{Set}}$$

$$\text{hollow}\,(\lambda\,\_\,\_) = \top \ \ \cdots$$

$$\frac{f\ g : \text{Obj}\ C[a,b] \qquad p : \text{hollow}\ f \qquad q : \text{hollow}\ g}{\text{coh}\ p\ q : \text{Obj}\ C[a,b][f,g]}$$

$$\text{hollow}\,(\text{coh}\,p\,q) = \top$$

Conclusions

- We have given a type-theoretic defition of weak ω-groupoids.
- And formalized it in Type Theory using the Agda system.
- This is the first step towards a weak ω-groupoid model of Type Theory
- Which can be used to give a computational interpretation of the univalence principle.