The Coherence Problem in HoTT

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Homotopy Type Theory and Univalent Foundations

based on joint work with Paolo Capriotti and Nicolai Kraus

From ITT to HoTT

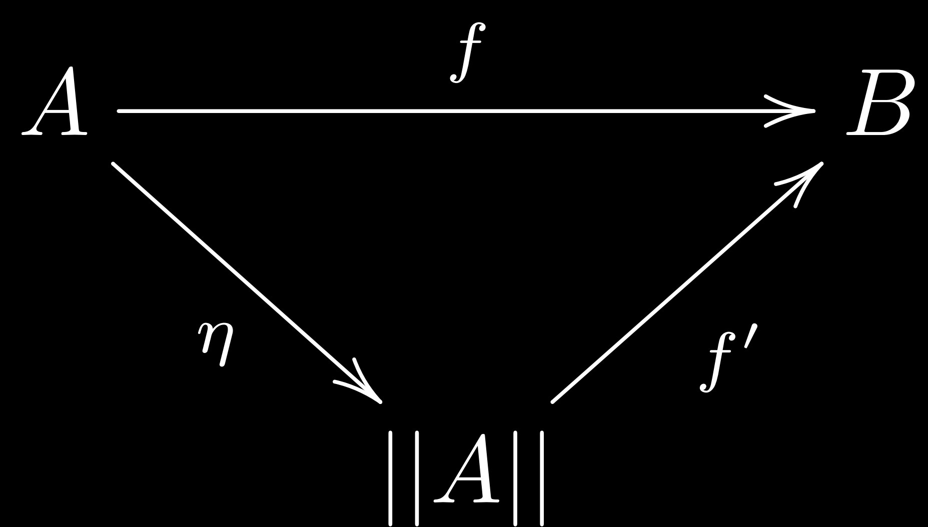
+ Univalence

- Uniqueness of Equality Proofs

Constant functions

$$f : A \to B \text{ with } B : \mathbf{Set}$$

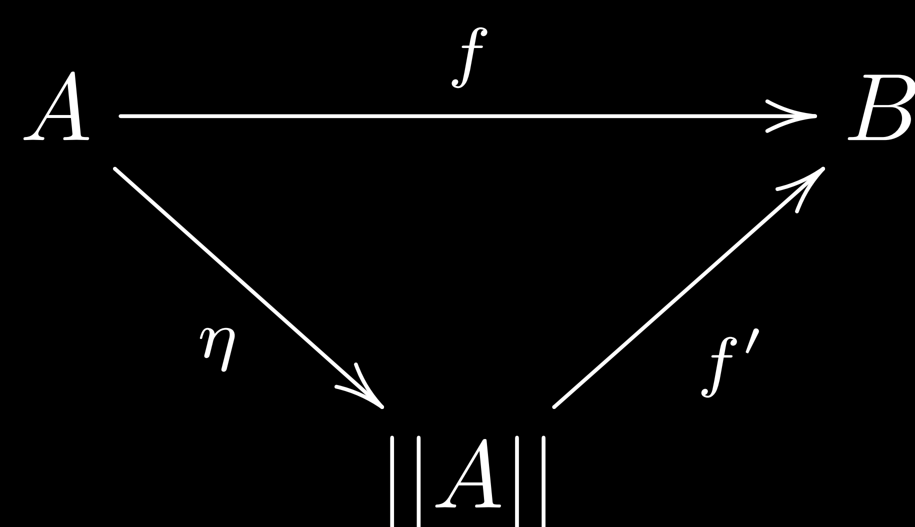
$$\text{const}(f) = \Pi_{a,a':A} f(a) = f(a')$$



Constant functions

$$f : A \to B \text{ with } B : \mathbf{Type}$$

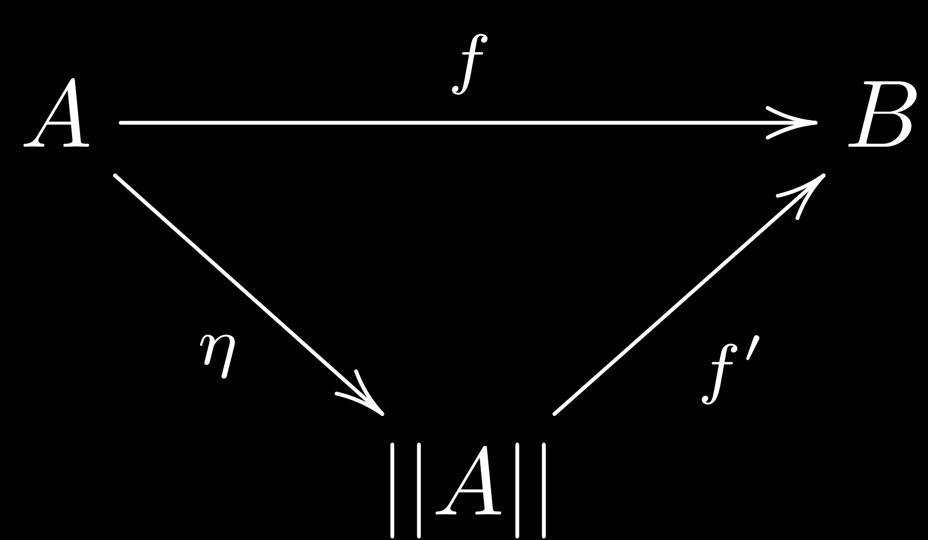
$$\text{const}(f) = ???$$



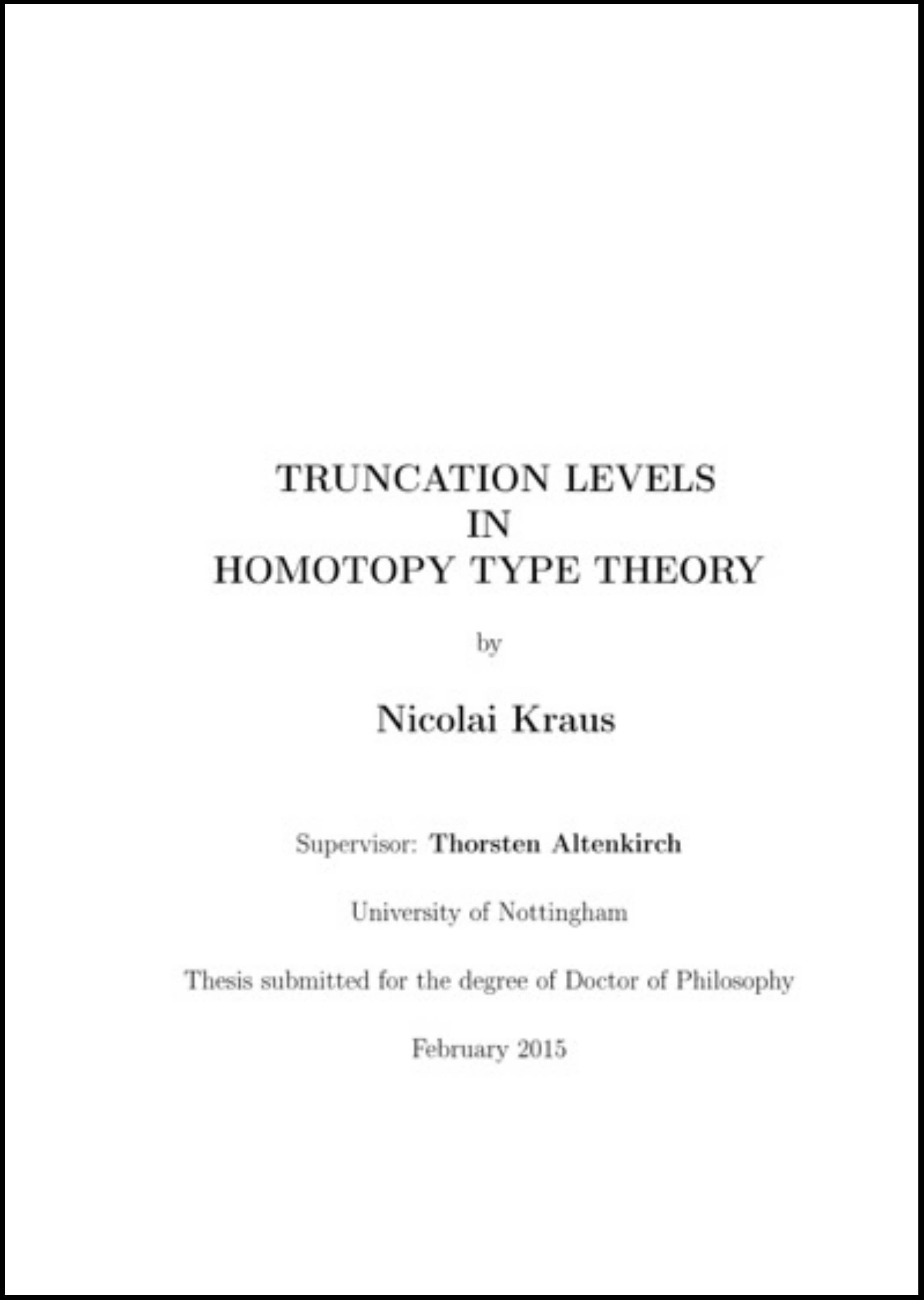
Constant functions

$$f : A \to B \text{ with } B : \mathbf{1-type}$$

$$\begin{aligned}\text{const}(f) = \Sigma c : \Pi_{a,a':A} f(a) = f(a'),\\ \Pi_{a,a',a'':A} c(a,a') \circ c(a',a'') = c(a,a'')\end{aligned}$$



Nicolai Kraus’ PhD



TRUNCATION LEVELS
IN
HOMOTOPY TYPE THEORY

by

Nicolai Kraus

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University of Nottingham

Thesis submitted for the degree of Doctor of Philosophy

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Semisimplicial Sets

$$
\begin{aligned}
\mathrm{SSset} = & \Sigma X_0 : \mathbf{Set} \\
& \Sigma X_1 : X_0 \to X_0 \to \mathbf{Set} \\
& \Sigma X_2 : \Pi_{x_0, x_1, x_2 : X_0} \\
& \qquad X_1 x_0 x_1 \to X_1 x_1 x_2 \to X_1 x_0 x_2 \to \mathbf{Set} \\
& \vdots
\end{aligned}
$$

Semisimplicial Sets

$$
\begin{aligned}
\text{SSset} = & \Sigma X : \Delta_i \to \mathbf{Set} \\
& \quad \Sigma X_m : \Pi_{i,j:\Delta_i} \Delta_i(i,j) \to X_j \to X_i \\
& \qquad X_m(\text{id}) = \text{id} \\
& \qquad \times X_m(f \circ g) = X_m(g) \circ X_m(f)
\end{aligned}
$$

Semisimplicial Types ?

$$
\begin{aligned}
\text{SStype} = & \Sigma X_0 : \mathbf{Type} \\
& \Sigma X_1 : X_0 \to X_0 \to \mathbf{Type} \\
& \Sigma X_2 : \Pi_{x_0, x_1, x_2 : X_0} \\
& \qquad X_1 x_0 x_1 \to X_1 x_1 x_2 \to X_1 x_0 x_2 \to \mathbf{Type} \\
& \vdots
\end{aligned}
$$

Semisimplicial Types ?

$$
\begin{aligned}
\text{SStype} = & \Sigma X : \Delta_i \to \mathbf{Type} \\
& \Sigma X_m : \Pi_{i,j:\Delta_i} \Delta_i(i,j) \to X_j \to X_i \\
& \Sigma \text{idl} : X_m(\text{id}) = \text{id} \\
& \Sigma \text{compl} : X_m(f \circ g) = X_m(g) \circ X_m(f) \\
& \vdots
\end{aligned}
$$

Conjecture

There is no way to define SStype in HoTT.

Proposal

- HoTT + strict equality ($\equiv$)
- Similar to HTS
- but based on ITT
- can be hacked into Agda

2 level HoTT

$\mathbf{type}$ — Pretypes

$\mathbf{Type}$ — Types

$$\mathrm{El} : \mathbf{Type} \to \mathbf{type}$$

$$\begin{aligned} \mathrm{El}('\mathbb{N}) &= \mathbb{N} \\ \mathrm{El}('\Pi x : A.B) &= \Pi x : \mathrm{El}(A).\mathrm{El}(B) \\ \mathrm{El}('\Sigma x : A.B) &= \Sigma x : \mathrm{El}(A).\mathrm{El}(B) \end{aligned}$$

2 level HoTT

$$\equiv : A \to A \to \mathbf{type}$$

strict equality

with J and K

$$= : A \to A \to \mathbf{Type} \qquad (A : \mathbf{Type})$$

univalent equality

Only J
Only eliminates into Type

Semisimplicial Types ?

$$\begin{aligned}\mathrm{SStype} =\, &\Sigma X : \Delta_i \to \mathbf{type}\\ &\Sigma X_m : \Pi_{i,j:\Delta_i} \Delta_i(i,j) \to X_j \to X_i\\ &\quad X_m(\mathrm{id}) \equiv \mathrm{id}\\ &\quad \times X_m(f \circ g) \equiv X_m(g) \circ X_m(f)\end{aligned}$$

$$\mathrm{SStype} : \mathbf{type}$$

Can we define

$$\mathrm{SStype} : \mathbf{Type}$$

?

Richard Garner

Truncated SStype

$$\begin{aligned} \mathrm{SS} &: \mathbb{N} \to \mathbf{Type} \\ \Phi &: \Pi_{j:\mathbb{N}} \mathrm{SS}(j) \to \mathbb{N} \to \mathbf{Type} \\ \Phi_m &: \Pi_{j:\mathbb{N}} \Pi_{\vec{X}:\mathrm{SS}(j)} \Delta_i(m,n) \to \Phi(j,\vec{X},n) \to \Phi(j,\vec{X},m) \end{aligned}$$

$$\Phi_m(X, \mathrm{id}) \equiv \mathrm{id}$$
$$\Phi_m(X, \mathrm{f} \circ \mathrm{g}) \equiv \Phi_m(X, \mathrm{g}) \circ \Phi_m(X, \mathrm{f})$$

Truncated SStype

$$\mathrm{SS} : \mathbb{N} \to \mathbf{Type}$$
$$\Phi : \Pi_{i:\mathbb{N}} \mathrm{SS}(j) \to \mathbb{N} \to \mathbf{Type}$$

$$\mathrm{SS}(0) = \mathbf{1}$$
$$\Phi(0, *, j) = 1$$

Truncated SStype

$$\mathrm{SS} : \mathbb{N} \to \mathbf{Type}$$

$$\Phi : \Pi_{i:\mathbb{N}} \mathrm{SS}(j) \to \mathbb{N} \to \mathbf{Type}$$

$$\mathrm{SS}(j+1) = \Sigma \vec{X} : \mathrm{SS}(j). \Phi(j, \vec{X}, j+1) \to \mathbf{Type}$$

$$\Phi(j+1, (\vec{X}, Y), m) = \Sigma \vec{x} : \Phi(i, \vec{X}, j).$$

$$\Pi \alpha : \Delta_i(m, j+1). Y(\Phi_m(i, \vec{X}, \alpha, \vec{x}))$$

$$\mathrm{SStype} : \mathbf{Type}$$

can now be obtained
as the Homotopy colimit of

$$\mathrm{SS}(n+1) \to \mathrm{SS}(n)$$

Thoughts

- Works for Reedy presheaves in general
- Exploit that $\mathbb{N}$ is fibrant
- Countermodel for SSType?
- Solve other coherence problems?