Normalization by evaluation for $\lambda^{\to 2}$

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Motivation

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- **Normalisation of evaluation** (NbE) used in the actual implementation of recent tools such as **Epigram**.
- Offers efficent implementations **and** straightforward correctness arguments.

More motivation

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- Here: Haskell as a poor man's type theory.
- In future: implementation within epigram.

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- $\lambda^{\rightarrow 2}$ without type-variables
  the simplest (interesting) typed λ-calculus!

$\lambda^{\rightarrow 2}$ in a nutshell

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$$\frac{}{\mathtt{True}, \mathtt{False} : \mathtt{Bool}} \qquad \frac{t : \mathtt{Bool} \quad u_0, u_1 : \sigma}{\mathtt{If}\, t\, u_0\, u_1 : \sigma}$$

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$$\begin{array}{llll}
\texttt{If}\,\texttt{True}\, u_0\, u_1 & =_{\beta\eta} & u_0 & (\beta) \\
\texttt{If}\,\texttt{False}\, u_0\, u_1 & =_{\beta\eta} & u_1 & (\beta) \\
\texttt{If}\, t\,\texttt{True}\,\texttt{False} & =_{\beta\eta} & t & (\eta) \\
f\,(\texttt{If}\, t\, u_0\, u_1) & =_{\beta\eta} & \texttt{If}\, t\,(f\, u_0)\,(f\, u_1) & (\xi)
\end{array}$$

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\end{array}$$

Categorically:

$$\mathrm{Hom}(\Gamma \times \mathrm{Bool}, \sigma) \simeq \mathrm{Hom}(\Gamma, \sigma) \times \mathrm{Hom}(\Gamma, \sigma)$$

Example

$$\begin{aligned}
\texttt{once} &= \lambda f^{\texttt{Bool}\to\texttt{Bool}} \lambda x^{\texttt{Bool}} f\, x \\
\texttt{twice} &= \lambda f^{\texttt{Bool}\to\texttt{Bool}} \lambda x^{\texttt{Bool}} f\, (f\, x) \\
\texttt{thrice} &= \lambda f^{\texttt{Bool}\to\texttt{Bool}} \lambda x^{\texttt{Bool}} f\, (f\, (f\, x))
\end{aligned}$$

$$\texttt{once}, \texttt{twice}, \texttt{thrice} : (\texttt{Bool} \to \texttt{Bool}) \to (\texttt{Bool} \to \texttt{Bool})$$

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$\texttt{once} =_{\beta\eta} \texttt{thrice}$

Why ?

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$$
\begin{aligned}
f\,(f\,(f\,\mathtt{True})) &=_{\beta\eta} \mathtt{If}\,(f\,\mathtt{True})\,(f\,(f\,\mathtt{True}))\,(f\,(f\,\mathtt{False}))\\
&=_{\beta\eta} \mathtt{If}\,(f\,\mathtt{True})\,\mathtt{True}\,(f\,(f\,\mathtt{False}))\\
&=_{\beta\eta} \mathtt{If}\,(f\,\mathtt{True})\,\mathtt{True}\,(\mathtt{If}\,(f\,\mathtt{False})\,(f\,\mathtt{True})\,(f\,\mathtt{False}))\\
&=_{\beta\eta} \mathtt{If}\,(f\,\mathtt{True})\,\mathtt{True}\,(\mathtt{If}\,(f\,\mathtt{False})\,\mathtt{False}\,\mathtt{False})\\
&=_{\beta\eta} \mathtt{If}\,(f\,\mathtt{True})\,\mathtt{True}\,\mathtt{False}\\
&=_{\beta\eta} f\,\mathtt{True}
\end{aligned}
$$

Symmetrically, we can show that $f\,(f\,(f\,\mathtt{False})) =_{\beta\eta} f\,\mathtt{False}$, and hence

$$
\begin{aligned}
&\mathtt{thrice}\\
&= \lambda f^{\mathtt{Bool}\to\mathtt{Bool}}\lambda x^{\mathtt{Bool}} f\,(f\,(f\,x))\\
&=_{\beta\eta} \lambda f^{\mathtt{Bool}\to\mathtt{Bool}}\lambda x^{\mathtt{Bool}} \mathtt{If}\,x\,(f\,(f\,(f\,\mathtt{True})))\,(f\,(f\,(f\,\mathtt{False})))\\
&=_{\beta\eta} \lambda f^{\mathtt{Bool}\to\mathtt{Bool}}\lambda x^{\mathtt{Bool}} \mathtt{If}\,x\,(f\,\mathtt{True})\,(f\,\mathtt{False})\\
&=_{\beta\eta} \lambda f^{\mathtt{Bool}\to\mathtt{Bool}}\lambda x^{\mathtt{Bool}} f\,x\\
&= \mathtt{once}
\end{aligned}
$$

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Why does this hold for $=_{\beta\eta}$?
Corollary of our NbE construction.

Another *corollary*: normalisation

```
Main> once
Lam (Bool :-> Bool) "f" (Lam Bool "x" (App (Var "f") (Var "x")))
Main> :t nf
nf :: Ty -> Tm -> Tm
Main> :t nf'
nf' :: Tm -> Maybe (Ty,Tm)
Main> nf' once
Just ((Bool :-> Bool) :-> (Bool :-> Bool),Lam (Bool :-> Bool) "x"
(If (App (Var "x") TTrue) (If (App (Var "x") TFalse) (Lam Bool "x"
 TTrue) (Lam Bool "x" (Var "x"))) (If (App (Var "x") TFalse) (Lam
Bool "x" (If (Var "x") TFalse TTrue)) (Lam Bool "x" TFalse))))
Main> nf' thrice
Just ((Bool :-> Bool) :-> (Bool :-> Bool),Lam (Bool :-> Bool) "x"
 (If (App (Var "x") TTrue) (If (App (Var "x") TFalse) (Lam Bool "x"
 TTrue) (Lam Bool "x" (Var "x"))) (If (App (Var "x") TFalse) (Lam
 Bool "x" (If (Var "x") TFalse TTrue)) (Lam Bool "x" TFalse))))
```

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1. Define a semantic interpretation

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3. Now define

$$\mathsf{nf}^{\sigma}\,t = \mathsf{quote}^{\sigma}\,[\![t]\!]$$

nf

$$\frac{}{\mathsf{nf}\ t =_{\beta\eta} t} \qquad \frac{t =_{\beta\eta} u}{\mathsf{nf}\ t = \mathsf{nf}\ u}$$

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The effectiveness of nf is witnessed by an implementation in Haskell.

Implementation in Haskell

Haskell-types can only approximate the intended types, e.g.

$$\mathsf{nf} \in \Pi_{\sigma \in \mathsf{Ty}} \mathsf{Tm}\,\sigma \to \mathsf{Tm}\,\sigma$$

is implemented as

```
nf :: Ty -> Tm -> Tm
```

The semantics

$$
\begin{aligned}
[\![\texttt{Bool}]\!] &= \mathrm{Bool} \\
&= \{\mathrm{true}, \mathrm{false}\} \\
[\![\sigma \to \tau]\!] &= [\![\sigma]\!] \to [\![\tau]\!]
\end{aligned}
$$

Implementation in Haskell

```
data El = STrue | SFalse | SLam Ty (El -> El)
```

Decision trees

Decision trees

- We use decision trees to enumerate types.

$$\frac{\sigma \in \mathsf{Ty}}{\mathsf{Tree}\,\sigma \in \star}\ \text{where}\ \frac{x \in [\![\sigma]\!]}{\mathsf{Val}\,x \in \mathsf{Tree}\,\sigma} \quad \frac{l, r \in \mathsf{Tree}\,\sigma}{\mathsf{Choice}\,l\,r \in \mathsf{Tree}\,\sigma}$$

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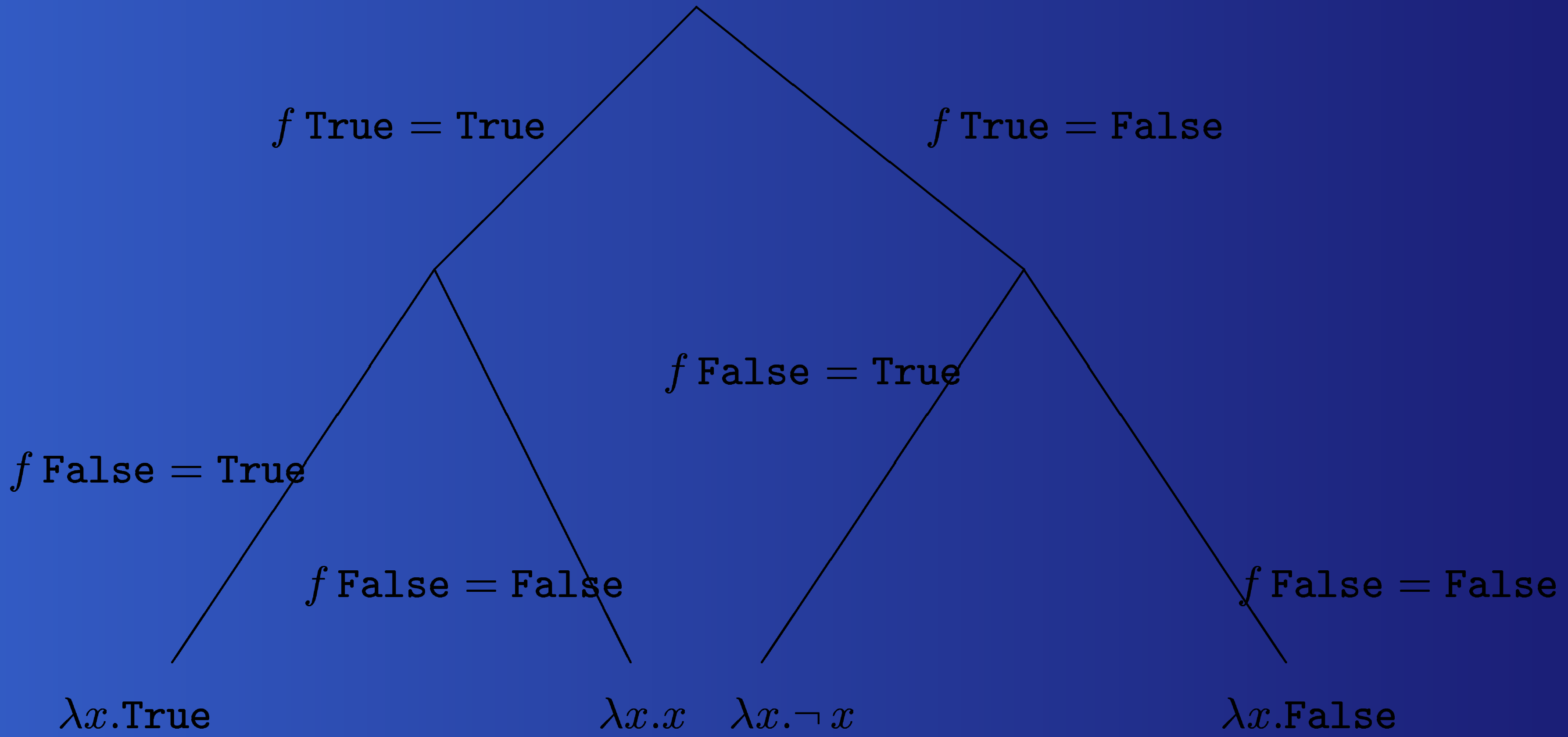
- We define by simultanous recursion over $\sigma \in \mathsf{Ty}$

$$\begin{array}{rcl} \mathsf{enum}\,\sigma & \in & \mathsf{Tree}\,\sigma \\ \mathsf{questions}\,\sigma & \in & [\![\sigma]\!] \to [\![\mathtt{Bool}]\!] \end{array}$$

Decision tree for $\texttt{Bool} \to \texttt{Bool}$



find

- We also implement

$$\frac{as \in [\llbracket \mathtt{Bool} \rrbracket] \qquad ts \in \mathsf{Tree}\,\llbracket\sigma\rrbracket \qquad as \diamond ts}{\mathsf{find}\, as\; ts \in \llbracket\sigma\rrbracket}$$

where $as \diamond ts$ expresses that the length of as matches the depth of ts.

Implementing quote

$$\frac{x \in [\![\sigma]\!]}{\mathsf{quote}^{\sigma} \in \mathsf{Tm}\,\sigma}$$

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\mathsf{quote}^{\sigma\to\tau}\,f &= \lambda\mathsf{x}^{\sigma}.\mathsf{find}_{\mathsf{syn}}\,[q\,\mathtt{x} \mid q \leftarrow \mathsf{questions}_{\mathsf{syn}}\,\sigma] \\
&\qquad (\mathsf{fmap}\,((\mathsf{quote}\,\tau).f)\,(\mathsf{enum}_{\mathsf{set}}\,\sigma))
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\end{aligned}$$

Note that we need only one bound variable!

Correctness of quote

How do we show

$$\frac{t : \sigma}{\mathsf{quote}^{\sigma}\,[\![t]\!] =_{\beta\eta} t}?$$

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Main lemma: $\dfrac{tR^\sigma x}{t =_{\beta\eta} \mathsf{quote}\, x}$

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- Extend the construction to $\lambda^{\rightarrow 01+\times}$ (*almost done*).

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  Useful as a hardware description language

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- Extend the construction to $\lambda^{\to 01+\times}$ (*almost done*).
- Extend the construction to $\lambda^{\Pi\Sigma 012}$ (finite Type Theory)
  Useful as a hardware description language
- Use BDDs instead of Decision Trees to improve efficiency.
- Can we extend this approach to type variables?

Related Work

- Neil Ghani
  Adjoint Rewriting
  PhD, 1995
- Thorsten Altenkirch,Peter Dybjer,
  Martin Hofmann,Phil Scott
  Normalization by evaluation for typed lambda calculus with coproducts
  LICS 2001
- Vincent Balat
  Une étude des sommes fortes : isomorphismes et formes normales
  PhD thesis, 2002