Why cubical type theory?

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Intensional Type Theory

- Basis of many current implementations:
  Coq, Agda, Idris
- *Inductive* definition of equality

$$\mathrm{refl} : \Pi x : A.x = x$$

- Propositional equality reflects definitional equality:
  If $\vdash p : a = b$ then $a \equiv b$.
- We cannot prove:

$$\lambda n.0 + n = \lambda n.n + 0$$

- However, we can't prove:

$$\lambda n.0 + n \neq \lambda n.n + 0$$

either!

Extensional vs Intensional Equality

Intensional equality

Two objects are equal iff they are constructed the same way.

Extensional equality

Two objects are equal iff they behave the same way.

- Extensional view essential for mathematical abstraction.

Homotopy Type Theory

- Higher dimensional view of propositional equality.
- Equality types expose this structure
- Univalence: Equality of types is equivalence (coherent isomorphism).
- Very extensional view of equality!
- Current *implementations*:
  introduce univalence as a postulate.
- Computationally and conceptually unsatisfactory.

Plan of action

- Homotopy Type Theory teaches us that equality can be described individually for each type former, eg.:

$$
\begin{array}{llcl}
\text{natural numbers:} & (\mathsf{zero} =_{\mathbb{N}} \mathsf{zero}) & \simeq & 1 \\
 & (\mathsf{zero} =_{\mathbb{N}} \mathsf{suc}\, m) & \simeq & 0 \\
 & (\mathsf{suc}\, m =_{\mathbb{N}} \mathsf{zero}) & \simeq & 0 \\
 & (\mathsf{suc}\, m =_{\mathbb{N}} \mathsf{suc}\, n) & \simeq & (m =_{\mathbb{N}} n) \\
\text{pairs:} & ((a, b) =_{A \times B} (a', b')) & \simeq & (a =_A a' \times b =_B b') \\
\text{functions:} & (f =_{A \to B} g) & \simeq & \\
 & \multicolumn{3}{l}{(\Pi(x, y : A).x =_A y \to f\, x =_B g\, y)} \\
\text{types:} & (A =_{\mathsf{U}} B) & \simeq & (A \simeq B)
\end{array}
$$

- Let's define equality separately for each type former, as above!

We need a heterogeneous equality

- The reason is type dependency
- Dependent pairs – the equality of the second components depends on the equality of the first components, eg.:

$$((m, xs) =_{\Sigma(i:\mathbb{N}).\mathit{Vec}\, i} (n, ys)) \simeq (\Sigma(r : m =_{\mathbb{N}} n).r \vdash xs =_{\mathit{Vec}\,-} ys)$$

- We add a heterogeneous equality:

$$\frac{a : A \quad b : B \quad e : A =_{\mathsf{U}} B}{a \sim_e b : \mathsf{U}}$$

$$\frac{xs : \mathit{Vec}\, m \quad ys : \mathit{Vec}\, n \quad \dfrac{r : m =_{\mathbb{N}} n}{\mathrm{ap}\, \mathit{Vec}\, r : \mathit{Vec}\, m =_{\mathsf{U}} \mathit{Vec}\, n}}{xs \sim_{\mathrm{ap}\, \mathit{Vec}\, r} ys : \mathsf{U}}$$

Heterogeneous equality (i)

- Specification:

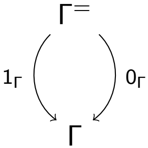
$$\frac{\Gamma \vdash}{\Gamma^{=} \vdash} \qquad \frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^{=} \vdash \sim_A : A[0] \to A[1] \to \mathsf{U}} \qquad 0_\Gamma, 1_\Gamma : \Gamma^{=} \Rightarrow \Gamma$$

- The operation $-^{=}$:

$$\begin{aligned} \emptyset^{=} &\equiv \emptyset \\ (\Gamma, x : A)^{=} &\equiv \Gamma^{=}, x_0 : A[0_\Gamma], x_1 : A[1_\Gamma], x_2 : x_0 \sim_A x_1 \end{aligned}$$

- Substitutions 0, 1 project out the corresponding components:

$$\begin{aligned} i_\emptyset &\equiv () &&: \emptyset \Rightarrow \emptyset \\ i_{\Gamma,A} &\equiv (i_\Gamma, x \mapsto x_i) &&: (\Gamma, x : A)^{=} \Rightarrow \Gamma, x : A \end{aligned}$$



Heterogeneous equality (ii)

Heterogeneous equality type defined as in "Plan of action":

$$\frac{\Gamma \vdash A : \mathsf{U}}{\Gamma^{=} \vdash \sim_A : A[0] \to A[1] \to \mathsf{U}}$$

$$f_0 \sim_{\Pi(x:A).B} f_1 \equiv \Pi(x_0 : A[0], x_1 : A[1], x_2 : x_0 \sim_A x_1). f_0\, x_0 \sim_B f_1\, x_1$$

$$(a, b) \sim_{\Sigma(x:A).B} (a', b') \equiv \Sigma(x_2 : a \sim_A a'). b \sim_B [x_0 \mapsto a, x_1 \mapsto a']\, b'$$

$$A \sim_{\mathsf{U}} B \equiv A \to B \to \mathsf{U} \text{ (parametricity)}$$

$$A \sim_{\mathsf{U}} B \equiv A \simeq B \text{ (later)}$$

$-^{=}$ is an endofunctor on the category of contexts

- Action on substitutions:

$$
\begin{aligned}
()^{=} &\equiv () \\
(\rho, x \mapsto t)^{=} &\equiv (\rho^{=}, x_0 \mapsto t[0], x_1 \mapsto t[1], x_2 \mapsto t^*)
\end{aligned}
$$

- Terms respect this equality (Reynold's abstraction theorem):

$$
\frac{\Gamma \vdash t : A}{\Gamma^{=} \vdash t^* : t[0] \sim_A t[1]}
$$

$$
\begin{aligned}
(f\ u)^* &\equiv f^*\ u[0]\ u[1]\ u^* \\
(\lambda x.t)^* &\equiv \lambda x_0, x_1, x_2\ .\ t^* \\
x^* &\equiv x_2 \\
\mathsf{U}^* &\equiv \sim_{\mathsf{U}}
\end{aligned}
$$

Homogeneous equality

- Heterogeneous equality:

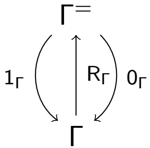
$$\frac{\Gamma \vdash A : \mathrm{U}}{\Gamma^{=} \vdash \sim_A : A[0] \to A[1] \to \mathrm{U}}$$

- We need equality in the same context:

$$\frac{\Gamma \vdash A : \mathrm{U}}{\Gamma \vdash =_A : A \to A \to \mathrm{U}}$$

- Therefore we define a substitution $R_\Gamma : \Gamma \Rightarrow \Gamma^{=}$:

$$\begin{aligned} \mathrm{R}_{\emptyset} &\equiv () &&: \emptyset \Rightarrow \emptyset \\ \mathrm{R}_{\Gamma.x:A} &\equiv (\mathrm{R}_\Gamma, x, x, \mathrm{refl}\, x) &&: (\Gamma.x : A) \Rightarrow (\Gamma.x : A)^{=} \end{aligned}$$



$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{refl}\, a \equiv (a^*)[\mathrm{R}_\Gamma] : \underbrace{a \sim_A [\mathrm{R}_\Gamma]\, a}_{a =_A a}}$$

What is $(\mathsf{refl}\,x)^*$? (i)

Maybe we could define it just as $\mathsf{refl}\,x^*$.

$$
\begin{aligned}
(x : A)^{=} \vdash (\mathsf{refl}\,x)^* &: \mathsf{refl}\,x_0 \sim_{(x \sim_{\mathsf{refl}\,A} x)^*} \mathsf{refl}\,x_1 \\
&\equiv \sim_{((\sim_{A^*[\mathsf{R}]})^*\,x_0\,x_1\,x_2\,x_0\,x_1\,x_2)} (\mathsf{refl}\,x_0)\,(\mathsf{refl}\,x_1) \\
(x : A)^{=} \vdash \mathsf{refl}\,x^* &: x_2 \sim_{\mathsf{refl}\,(x_0 \sim_{A^*} x_1)} x_2 \\
&\equiv \sim_{((\sim_{A^*[\mathsf{R}]})^*\,x_0\,x_0\,(\mathsf{refl}\,x_0)\,x_1\,x_1\,(\mathsf{refl}\,x_1))} x_2\,x_2
\end{aligned}
$$

Higher dimensions

By iterating $-^{=}$, we get higher dimensional cubes. Eg. if $A : \mathrm{U}$, elements of $x : A$, $(x : A)^{=}$, $((x : A)^{=})^{=}$, $(x : A)^{3}$ look like this:

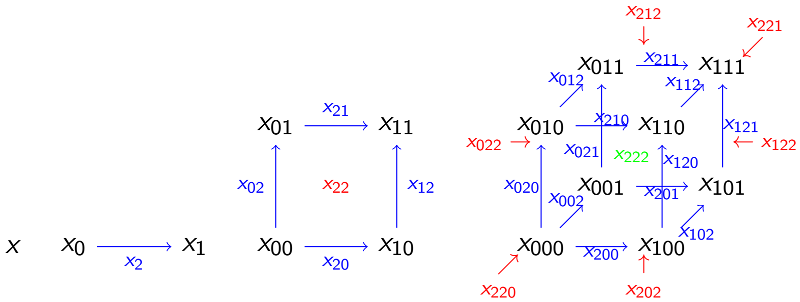
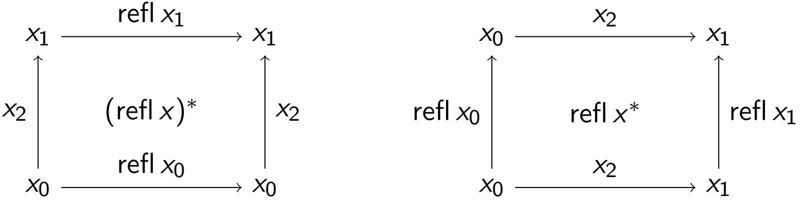


Figure: Cubes of dimension 0-3.

What is $(\mathsf{refl}\,x)^*$? (ii)

$$(x : A)^= \vdash (\mathsf{refl}\,x)^* : \mathsf{refl}\,x_0 \sim_{(x \sim_{\mathsf{refl}\,A} x)^*} \mathsf{refl}\,x_1$$
$$\equiv \sim_{((\sim_{A^*[\mathsf{R}]})^*\,x_0\,x_1\,x_2\,x_0\,x_1\,x_2)} (\mathsf{refl}\,x_0)\,(\mathsf{refl}\,x_1)$$
$$(x : A)^= \vdash \mathsf{refl}\,x^* : x_2 \sim_{\mathsf{refl}\,(x_0 \sim_{A^*} x_1)} x_2$$
$$\equiv \sim_{((\sim_{A^*[\mathsf{R}]})^*\,x_0\,x_0\,(\mathsf{refl}\,x_0)\,x_1\,x_1\,(\mathsf{refl}\,x_1))} x_2\,x_2$$



If we swap the vertical and horizontal dimensions we get one from the other.

Swap

We define a substitution $\mathsf{S}_\Gamma : \Gamma^2 \Rightarrow \Gamma^2$.
Visually:

$$
\begin{array}{ccc}
x_{01} & \xrightarrow{x_{21}} & x_{11} \\
x_{02} \uparrow & x_{22} & \uparrow x_{12} \\
x_{00} & \xrightarrow{x_{20}} & x_{10}
\end{array}
\quad \stackrel{\mathsf{S}_{x:A}}{\longmapsto} \quad
\begin{array}{ccc}
x_{10} & \xrightarrow{x_{12}} & x_{11} \\
x_{20} \uparrow & x_{22}[\mathsf{S}_{x:A}] & \uparrow x_{21} \\
x_{00} & \xrightarrow{x_{02}} & x_{01}
\end{array}
$$

Now we can say that

$$(\mathsf{refl}\, x)^* \equiv (x_2[\mathsf{R}])^* \equiv x_{22}[(\mathsf{R}_{x:A})^=] \equiv x_{22}[\mathsf{S}_{x:A}\mathsf{R}_{x:A}]$$

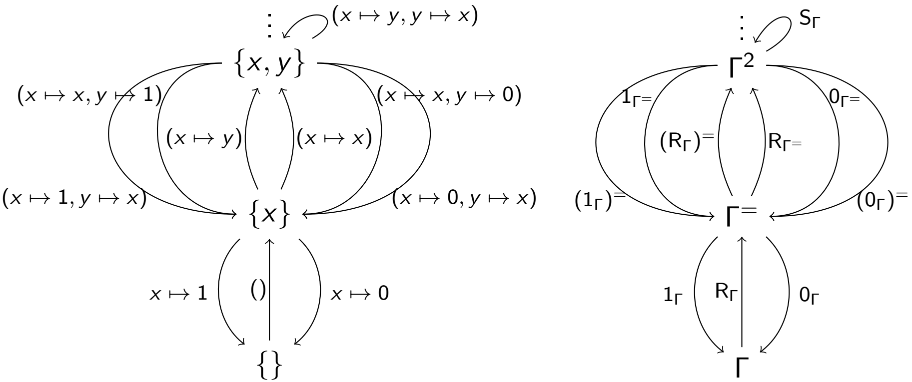
Semantics

- We can interpret our types as (covariant) presheaves over $\mathcal{C}$:

  objects finite sets

  morphisms $\mathcal{C}(I, J)$ is given by a function $f : I \to J + \{0, 1\}$ that is injective on J, that is

$$f(i) = inl(j) = f(k) \to i = k$$



Equality?

- We capture internal relational parametricity not equality.
- Bernardy, Moulin: A computational interpretation of parametricity, 2012
- Need to restrict our interpretation to Kan cubical sets.
- How to do this syntactically?

Geometry over syntax

- Taking a homotopical view of this interpretation leads to a very elegant syntax.
- Cohen, Coquand, Huber, Mörtberg:
  Cubical Type Theory: a constructive interpretation of the univalence axiom