Containers in Homotopy Type Theory

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- Univalence: isomorphic types are equal.
- Hence equality cannot be proof-irrelevant.
- Kraus: we can construct types with arbitrary complex equality (*n*-types) using universes + univalence.
- Higher inductive types (HITs) give us an alternative way to construct *n*-types without using universes.
- Main application: synthetic homotopy theory (e.g. define the *n*-spheres and verify their properties).
- Today: applications of HITs to datatypes.

Container

- Container = polynomial functor
- W-types = initial algebras of containers
- Given by

  Shapes S : **Set**

  Positions $P : S \to$ **Set**

  we write $S \lhd P$.
- Extension as a functor:

$$[\![S \lhd P]\!] : \mathbf{Set} \to \mathbf{Set}$$

$$[\![S \lhd P]\!](A) = \Sigma s : S.P(s) \to A$$

- Examples ($\mathrm{Fin}(n) = \{0, 1, \ldots, n-1\}$)

$$[\![1 \lhd 1]\!](A) = A$$

$$[\![1 \lhd \mathrm{Fin}(n)]\!](A) = A^n$$

$$[\![n : \mathbb{N} \lhd \mathrm{Fin}(n)]\!](A) = \mathrm{List}(A)$$

$$[\![1 \lhd \mathbb{N}]\!](A) = \mathrm{Stream}(A)$$

Constructions on containers

Coproducts

$$(S \lhd P) + (T \lhd Q) = x : S + T \lhd \text{case } x \text{ of } \begin{cases} \text{inl}(s) \to P(s) \\ \text{inr}(t) \to Q(t) \end{cases}$$

Products

$$(S \lhd P) \times (T \lhd Q) = (s, t) : S \times T \lhd P(s) + Q(t)$$

Composition

$$(S \lhd P) \circ (T \lhd Q) = \Sigma s : S, f : P(s) \to T \lhd \Sigma p : P(s).Q(f(p))$$

$$\begin{aligned} [\![(S \lhd P) + (T \lhd Q)]\!](A) &= [\![S \lhd P]\!](A) + [\![T \lhd Q]\!](A) \\ [\![(S \lhd P) \times (T \lhd Q)]\!](A) &= [\![S \lhd P]\!](A) \times [\![T \lhd Q]\!](A) \\ [\![(S \lhd P) \circ (T \lhd Q)]\!](A) &= [\![S \lhd P]\!]([\![T \lhd Q]\!](A)) \end{aligned}$$

Container morphisms

Given $(S \lhd P), (T \lhd Q)$ a morphism $f \lhd r$ is given by

$$\begin{aligned} f &: S \to T \\ r &: \Pi_{s:S} Q(f(s)) \to P(s) \end{aligned}$$

whose extension is a natural transformation given by

$$\begin{aligned} &[\![f \lhd r]\!] : \Pi_{A:\mathbf{Set}} [\![S \lhd P]\!](A) \to [\![T \lhd Q]\!](A) \\ &[\![f \lhd r]\!](A, (s, \vec{a})) = (f(s), \vec{a} \circ r(s)) \end{aligned}$$

Read $\Pi_{A:\mathbf{Set}} T(A)$ as $\int_{A:\mathbf{Set}} T(A)$.

Container morphisms

Examples:

tail

$$\text{tail} : \Pi_{A:\mathbf{Set}} \text{List}(A) \to \text{List}(A)$$
$$\text{tail}([a_0, a_1, \ldots, a_n]) = [a_1 \ldots a_n]$$

$$\text{tail} = \lambda n.n \dot{-} 1 \ \lhd \ \lambda n, i.i - 1$$

reverse

$$\text{reverse} : \Pi_{A:\mathbf{Set}} \text{List}(A) \to \text{List}(A)$$
$$\text{reverse}([a_0, a_1, \ldots, a_n]) = [a_n, a_{n-1} \ldots a_0]$$

$$\text{reverse} = \lambda n.n \ \lhd \ \lambda n, i.n - i$$

Container morphisms

Given a container $S \lhd P$ and a functor $F : \mathbf{Set} \to \mathbf{Set}$:

$$\begin{aligned}
&\Pi_{A:\mathbf{Set}}[\![S \lhd P]\!](A) \to F(A) \\
&= \Pi_{A:\mathbf{Set}}(\Sigma_{s:S}P(s) \to A) \to F(A) \\
&= \Pi_{A:\mathbf{Set}}\Pi_{s:S}(P(s) \to A) \to F(A) \\
&= \Pi_{s:S}\Pi_{A:\mathbf{Set}}(P(s) \to A) \to F(A) \\
&= \Pi_{s:S}F(P(s)) \qquad (\mathit{Yoneda})
\end{aligned}$$

Let $F = [\![T \lhd Q]\!]$ then

$$\begin{aligned}
&\Pi_{A:\mathbf{Set}}[\![S \lhd P]\!](A) \to [\![T \lhd Q]\!](A) \\
&= \Pi_{s:S}[\![T \lhd Q]\!](P(s)) \\
&= \Pi_{s:S}\Sigma_{t:T}Q(t) \to P(s) \\
&= \Sigma_{f:S \to T}Q(f(s)) \to P(s)
\end{aligned}$$

Hence the functor $[\![-]\!] : \mathrm{Cont} \to (\mathbf{Set} \to \mathbf{Set})$ is full and faithful.

Multisets as containers?

- Multisets are lists quotiented by permutations, i.e.

$$\{a, a, b\}^M = \{b, a, a\}^M \qquad \{a, a, b\}^M \neq \{a, b\}^M$$

- $$\mathrm{M}(A) = \mathrm{List}(A)/\sim$$
  $$l \sim l' = l \text{ is a permutation of } l'$$

- We can show that all container preserve pullbacks, but M does not preserve pullbacks.
- Multisets are not representable as containers in conventional Type Theory.

Quotient containers / Symmetric containers

Abbot, A., Ghani, McBride MPC 2004
Constructing Polymorphic Programs with Quotient Types
Uses quotient types to represent containers with permutable positions.

Gylterud MSc thesis 2012
Symmetric containers
Generalizes containers by replacing the set of positions by a groupoid.

Multisets in HoTT

- We can can define the following HIT:

$$S_M : \textbf{Type}_1$$
$$e : \mathbb{N} \to S_M$$
$$\epsilon : \mathrm{Fin}(m) = \mathrm{Fin}(n) \to e(m) = e(n)$$

- And the following family :

$$P_M : S_M \to \textbf{Set}$$
$$P_M(e(n)) = \mathrm{Fin}(n) \qquad P_M(\epsilon(\alpha)) = \mathrm{transport}(Fin, \alpha)$$

- $M = S_M \lhd P_M$ is the multiset container, that is $[\![M]\!](A)$ is the set of multisets over A.

Cycles

- Another example are cycles, i.e. lists quotiented by rotations. E.g.

$$\{a, b, c\}^C = \{c, a, b\}^C \qquad \{a, b, c\}^C \neq \{a, c, b\}^C$$

- We define the HIT

$$S_C : \mathbf{Type}_1$$
$$e : \mathbb{N} \to S_C$$
$$\epsilon : \mathrm{Fin}(m) \to e(m) = e(m) \qquad \delta : \epsilon(0) = \mathrm{refl}$$

- And the following family :

$$P_C : S_C \to \mathbf{Set}$$
$$P_C(e(n)) = \mathrm{Fin}(n) \qquad P_C(\epsilon(i)) = \lambda j.i + j \bmod(n)$$

- $C = S_C \lhd P_C$ s.t.
  $[\![C]\!](A)$ is the set of cycles over A.

Main insight

In HoTT quotient containers become ordinary containers, if we allow S : **Type** (that is not necessarily a set).

Zipper

- The notion of the derivative of a parametric datatype was introduced by Conor McBride.
- Conor was generalizing the notion of a zipper, introduced by Gerard Huet.
- A zipper is a datastructure which represents a position within a tree.

Zipper

- For example a zipper for binary trees

$$T = 1 + T^2$$

is

$$Z = 1 + 2 \times T \times Z$$

- In general given a datatype

$$T = F(T)$$

the corresponding zipper is given by

$$Z = 1 + \partial F(Z)$$

Derivatives

- Given

$$F : \mathbf{Set} \to \mathbf{Set},$$

$$\partial F : \mathbf{Set} \to \mathbf{Set}$$

is the type of *one hole contexts*.

- Conor noticed that this operation satisfies the laws of differential calculus, e.g.

$$\begin{aligned}
\partial(F + G)(A) &= \partial F(A) + \partial G(A) \\
\partial(F \times G)(A) &= \partial F(A) \times G(A) + F(A) \times \partial G(A) \\
\partial(F \circ G)(A) &= \partial F(G(A)) \times \partial G(A)
\end{aligned}$$

Cartesian morphisms

- To formally specify derivatives of containers, we need cartesian morphisms of containers.
- Cartesian morphisms do neither forget or copy data.
- Given $(S \lhd P), (T \lhd Q)$ a cartesian morphism $f \lhd \phi$ is given by

$$\begin{aligned} f &: S \to T \\ \phi &: \Pi_{s:S} Q(f(s)) = P(s) \end{aligned}$$

- Each cartesian morphism induces an ordinary morphism by transporting along $\phi(s)$.
- Indeed it's extension are exactly the natural transformations whose naturality squares are pullbacks.

Specifying ∂

- Using cartesian morphisms, we can specify ∂.

$$\mathrm{Cart}(K \times I, F) = \mathrm{Cart}(K, \partial F)$$

  where $I = 1 \ \lhd \ 1$ and K is any container.
- This is the translation of the intuitive idea of a one-hole context.

Explicit definition of ∂

- Give A : **Type**, $a : A$ we can specify $A - a$: **Type** as satisfying

$$(A = 1 + B) = (\Sigma a : A.A - a = B)$$

- $A - a$ exists, iff equality on A is decidable and then it is given as

$$A - a :\equiv \Sigma a' : A.a \neq a'$$

- We can show that $\partial(S \lhd P)$ exists, iff for all $s : A$, equality on $P(s)$ is decidable and then it is given as

$$\partial(S \lhd P) :\equiv \Sigma s : S, p : P(s) \lhd P(s) - p$$

Examples

- $\partial(\lambda A.A^{n+1}) = \lambda A.\mathrm{Fin}(n+1) \times A^n$

$$\begin{aligned} &\partial(1 \ \triangleleft\ \mathrm{Fin}(n+1)) \\ &= \ i : \mathrm{Fin}(n+1) \ \triangleleft\ \mathrm{Fin}(n+1) - i \\ &= \ \mathrm{Fin}(n+1) \ \triangleleft\ \mathrm{Fin}(n) \end{aligned}$$

- What is $\partial\mathrm{List}$?
- $\partial\mathrm{List} = \mathrm{List}^2$

$$\begin{aligned} &\partial\mathrm{List} \\ &= \ \partial(n : \mathbb{N} \ \triangleleft\ \mathrm{Fin}(n)) \\ &= \ \Sigma n : \mathbb{N}. i : \mathrm{Fin}(n) \ \triangleleft\ \mathrm{Fin}(n) - i \\ &= \ (l, m) : \mathbb{N} \times \mathbb{N} \ \triangleleft\ \mathrm{Fin}(l) + \mathrm{Fin}(m) \\ &= \ (l : \mathbb{N} \ \triangleleft\ \mathrm{Fin}(l)) \times (m : \mathbb{N} \ \triangleleft\ \mathrm{Fin}(m)) \\ &= \ \mathrm{List}^2 \end{aligned}$$

Examples

- What is the derivative of multisets $\partial\mathrm{M}$?
- $\partial\mathrm{M} = \mathrm{M}$!

$$
\begin{aligned}
&\partial\mathrm{M} \\
&\quad = \partial(S_M \triangleleft P_M) \\
&\quad = \Sigma e(n) : S_M.i : P_M(n) \triangleleft P_M(n) - i \\
&\quad = S_M \triangleleft P_M(n)
\end{aligned}
$$

Antiderivatives

- Gylterud asked wether antiderivatives exist.
- He noticed that we don't have antiderivatives in general if we rely on S : **Set**.
- Eg. there are no antiderivatives of $F(A) = A^n$ and hence there is no anti-derivative of List.
- However, this is different in the presence of HITs.
- What is the antiderivative of List?
- $\partial \mathrm{C} = \mathrm{List}$

$$
\begin{aligned}
\partial \mathrm{C} &= \partial(S_C \lhd P_C) \\
&= \Sigma e(n) : S_C.i : P_C(n) \lhd P_C(n) - i \\
&= m : \mathbb{N} \lhd \mathrm{Fin}(n) \\
&= \mathrm{List}
\end{aligned}
$$

Analytic containers

- To each (discrete) container $S \lhd P$ we can associate the Taylor series:

$$T_{S \lhd P} : \mathbb{N} \to \textbf{Set}$$
$$T_{S \lhd P}(n) = \partial^n(S \lhd P)(0)/S_n$$

- A container is analytic, iff

$$[\![\Sigma_{n:\mathbb{N}} T_{S \lhd P}(n) \lhd \mathrm{Fin}(n)]\!] = [\![S \lhd P]\!]$$

- Gylterud: A discrete container $(S \lhd P)$ is analytic iff $P(s)$ is finite for all $s : S$.
- Gylterud: All analytic container have antiderivatives. The antiderivatives are given by a HIT whose elements are $\Sigma n : \mathbb{N}. T_{S \lhd P}(n)$ and the equality and positions as for the cycles.

Sattler's result

- Christian Sattler showed that a cycle of size n has an antiderivative iff there is a finite field of size $n + 1$.
- The derivative of this field is given by the cyclic group of bijective affine transformations on the field that fix 0:

$$\{x \mapsto ax \mid a : F, a \neq 0\}$$

- Hence there is no antiderivative of the cycle of size 5 (since there is no finite field of size 6).
- Hence there is no antiderivative of cycles in general.