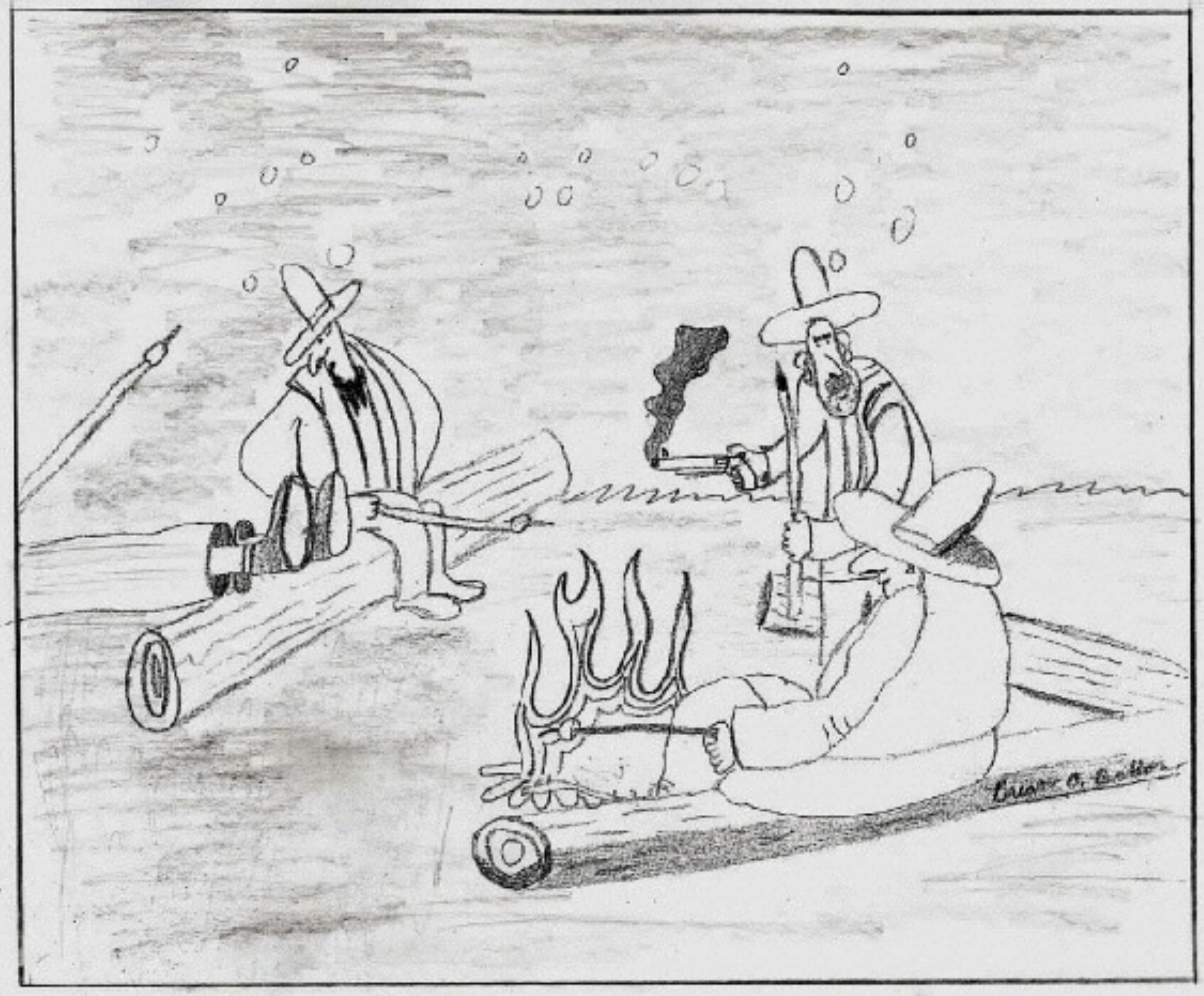
HoTT Christmas



You guys are both my witnesses... He insinuated that ZFC set theory is superior to Type Theory!

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How do we teach Mathematics?

- Use informal set theory?
- Definition

$$A \cap B := \{x \mid x \in A \wedge x \in B\}$$

- But what is

$$\mathbb{N} \cap \mathbb{B}$$

?

More stupid questions

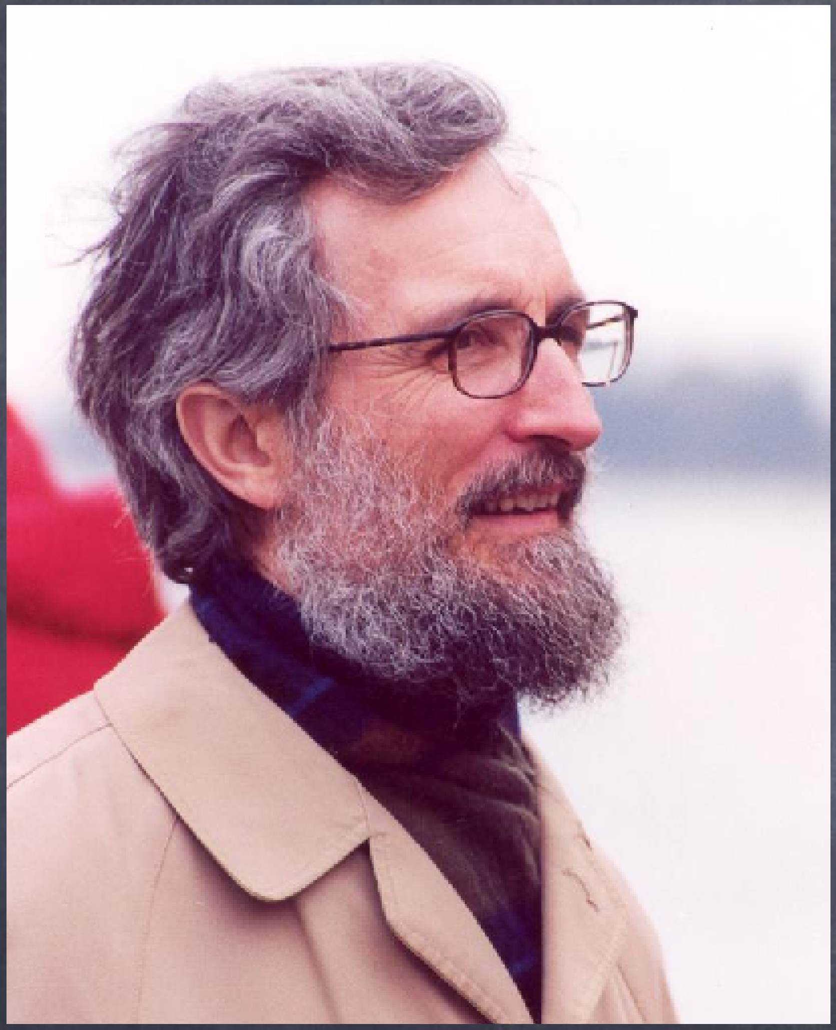
$$A \times B \subseteq \mathcal{P}(\mathcal{P}(A \cup B))?$$

$$A \to B \subseteq \mathcal{P}(A \times B)?$$

What is the problem?

- In set theory we can ask questions about the intensional properties of constructions like $\mathbb{N}, \mathbb{B}, \times, \rightarrow$
- Also their definitions seem quite arbitrary.
- This is a consequence of the idea that elements of sets exist independently of the set they inhabit.

The alternative



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Per Martin-Löf

Vladimir Voevodsky

= Homotopy Type Theory (HoTT)

Types come first!

- In Type Theory elements of a type do not exist in isolation of the type they inhabit!
- In Set Theory $a \in A$ is a proposition
in Type Theory $a : A$ is a judgment.
- We cannot define $A \cap B, A \cup B, A \subseteq B$ on arbitrary types.

Univalence

- Because we cannot talk about intensional properties of constructions ...
- ... all constructions are invariant under extensional equivalence.
- This is expressed formally by Voevodsky's univalence principle.

Type Theory for dummies

Constructions in Type Theory

$A \to B$	Functions special case of Π types
$A \times B$	Tuples special case of Σ types
$\mathbb{B}$	Bool, special case of a finite type
$\mathbb{N}$	natural numbers special case of a tree type
$a =_A b$	equality types
$\mathbf{Type}_i$	universes

Anatomy of a type

Formation	How to form a type?
Introduction	How to form elements?
Non-dependent elimination	How to define non-dependent functions from a type?
Dependent elimination	How to define dependent functions from a type?
Computation	How to compute?

Anatomy of a type

Formation	How to form a type?
Introduction	How to form elements?
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Example : tuples

Formation	If $A, B : \mathbf{Type}$ then $A \times B : \mathbf{Type}$

Example : tuples

Introduction	If $a : A, b : B$ then $(a, b) : A \times B$

Example : tuples

Non-dependent elimination	To define $f : A \times B \to C$ we need $g : A \to B \to C$
Computation	$f\,(a, b) \equiv g\,a\,b$

Example : tuples

Dependent elimination $C : A \times B \to \mathbf{Type}$	To define $f : \Pi p : A \times B . C\, p$ we need $g : \Pi a : A . \Pi b : B . C\,(a, b)$
Computation	$f\,(a, b) \equiv g\, a\, b$

Eliminator

- The dependent elimination principle can also be expressed by an eliminator

$$E_{A\times B} : \Pi_{C:A\times B\to\mathbf{Type}}\Pi_{g:\Pi a:A\Pi b:B.C\,(a,b)}\Pi p : A\times B.C\,p$$

- with the computation rule

$$E_{A\times B}\,C\,g\,(a,b) \equiv g\,a\,b$$

Propositions as types

- Using the idea to identify a proposition with the type of its proofs
- we can use dependent elimination to prove things.
- E.g. $\Pi p : A \times B . (\pi_1\, p, \pi_2\, p) = p.$
- where $\pi_i : A_1 \times A_2 \to A_i$ can be defined using non-dependent elimination

Canonicity

- The elimination principle makes sure that all functions applied to canonical elements can be eliminated.
- All closed terms of a type are computationally equal ($\equiv$) to a term built from constructors.

Equality for beginners

Example : equality

Formation	If $a, b : A$ then $a =_A b : \mathbf{Type}$

Example : equality

Introduction	If $a : A$ then $\mathrm{refl}\, a : a =_A a$

Example : equality

Non-dependent elimination	To define $f : \Pi x : A, a = x \to P\,x$ we need $g : P\,a$
Computation	$f\,a\,(\mathrm{refl}\,a) \equiv g$

Example : equality

Dependent elimination $P : \Pi x : A. a = x \to \mathbf{Type}$	To define $f : \Pi x : A, \Pi p : a = x \to P\, x\, p$ we need $g : P\, a\, (\mathrm{refl}\, a)$
Computation	$f\, a\, (\mathrm{refl}\, a) \equiv g$

The structure of equality types

- Using the elimination principle we can show that all types have the structure of a groupoid.

$$\begin{array}{rcl} \mathrm{refl} & : & \Pi a : A, a = a \\ (-)^{-1} & : & \Pi_{a,b:A}, a = b \to b = a \\ - \circ - & : & \Pi_{a,b,c:A} b = c \to a = b \to a = c \end{array}$$

$$\begin{array}{rcl} \lambda & : & \Pi_{a,b:A} \Pi p : a = b, p \circ (\mathrm{refl}\, a) = p \\ \rho & : & \Pi_{a,b:A} \Pi p : a = b, (\mathrm{refl}\, b) \circ p = p \\ \vdots & \vdots & \vdots \end{array}$$

- Each function gives rise to a functor: for $f : A \to B$ we have

$$f^{=} : \Pi_{a.a':A} a =_A a' \to f\, a = f\, a'$$

The structure of equality types

- Using the elimination principle we can show that all types have the structure of an ω-groupoid.

$$\begin{array}{rcl} \mathrm{refl} & : & \Pi a : A, a = a \\ (-)^{-1} & : & \Pi_{a,b:A}, a = b \to b = a \\ - \circ - & : & \Pi_{a,b,c:A} b = c \to a = b \to a = c \end{array} \qquad \begin{array}{rcl} \lambda & : & \Pi_{a,b:A} \Pi p : a = b, p \circ (\mathrm{refl}\, a) = p \\ \rho & : & \Pi_{a,b:A} \Pi p : a = b, (\mathrm{refl}\, b) \circ p = p \\ \vdots & \vdots & \vdots \end{array}$$

- Each function gives rise to an ω-functor: for $f : A \to B$ we have

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Univalence for cat lovers

Propositions

- We say that a type is a proposition (or a (-1)-type) if all elements are equal.
- Hence the only observable property of this type is wether it is inhabited.

Sets

- We say that a type is a set (or a 0-type) if all its equalities are propositions.
- In general we say that a type is an (n+1)-type if all its equalities are n-types

Univalence for propositions

- We define logical equivalence having functions in both directions.

$$A \iff B := \Sigma f : A \to B$$
$$g : B \to A$$

- Univalence for propositions implies that equality for propositions is logically equivalent to logical equivalence.

$$(A = B) \iff (A \iff B)$$

Univalence for sets

- Isomorphism is a refinement of logical equivalence: $A \simeq B :=$

$$\begin{aligned} \Sigma f &: A \to B \\ g &: B \to A \\ \eta &: \Pi a : A, g\,(f\,a) = a \\ \epsilon &: \Pi b : B, f\,(g\,b) = b \end{aligned}$$

- Univalence for sets implies that equality for sets is isomorphic to isomorphism:

$$(A = B) \simeq (A \simeq B)$$

Univalence for types

- Equivalence is a refinement of isomorphism:

$$A \cong B :=$$

$$\begin{aligned} &\Sigma f : A \to B \\ &\quad g : B \to A \\ &\quad \eta : \Pi a : A, g\,(f\,a) = a \\ &\quad \epsilon : \Pi b : B, f\,(g\,b) = b \\ &\quad \delta : \Pi a : A, f^{=}\,(\eta\,a) = \epsilon\,(f\,a) \end{aligned}$$

- Univalence implies that equality for types is equivalent to equivalence:

$$(A = B) \cong (A \cong B)$$

Canonicity ?

- We add univalence as a constant :

$$f : A = B \to A \cong B$$

$$\mathrm{uval} : \mathrm{isEquivalence}\, f$$

- However, this destroys the computational symmetry of introduction and elimination for equality types.

What I would have talked about to a more sophisticated audience

Cubical Type Theory

- We consider an alternative presentation of equality types where equality is defined as a logical relation.
- Since we have to deal with dependent types this we have to use heterogenous equality.
- This is related to internal parametricity ala Bernardy and Moulin...
- ...and Coquand & Huber's work on the constructive cubical set model.

Back to the future

How should we teach Mathematics?

- Use informal Type Theory!
- Encourages sensible use of Mathematics!
- Given $A, B : X \to \mathbf{Prop}$ define

$$A \cap B : X \to \mathbf{Prop}$$
$$(A \cap B)\, x = A\, x \wedge B\, x$$