Normalisation by Completeness

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How NBE (for $=^{\beta\eta}$) was discovered . . .

- Helmut Schwichtenberg needed to implement $\beta\eta$-conversion for his MINLOG system.
- The implementation language was SCHEME.
- He wondered how he could exploit SCHEME's evaluator. . .
- This lead to the LICS 91 paper by Berger and Schwichtenberg.

How NBE should have been discovered. . .

- Derive normalisation from intuitionistic completeness proofs.
- Simpler then NBE because we ignore equality.
- Minimal logic ($\approx$ simply typed λ calculus).
- Investigate disjunction ($\approx$ coproducts).

References:

CTCS 95 A.,Hofmann, Streicher
Reconstruction of a reduction-free normalisation proof

LICS 01 A.,Dybjer, Hofmann, Scott
Normalization by evaluation for typed lambda calculus with coproducts

Minimal Logic

$$\frac{}{\Gamma.A \vdash A} \qquad \frac{\Gamma \vdash A}{\Gamma.B \vdash A}$$

$$\frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B} \qquad \frac{\Gamma.A \vdash B}{\Gamma \vdash A \to B}$$

with:

Propositions $A :: X \mid \cdots \mid A \to A$ with $X = \{P, Q, R, \ldots\}$ atoms.

Contexts $\Gamma :: \mathrm{empty} \mid \Gamma.A$

Exercise

Show that $\nvdash (P \to P) \to P$.

Solution

Use truthtable semantics: if $\vdash A$ then $[\![A]\!]_\rho = \mathrm{true}$ for any truth assignment. However

$$[\![(P \to P) \to P]\!]_{P \mapsto \mathrm{false}} = \mathrm{false}$$

hence $\nvdash (P \to P) \to P$.

Exercise

Show that $\nvdash ((P \to Q) \to P) \to P$.

Solutions

1. Use Normalisation...
2. Use Kripke semantics...

Normal derivations

$$\frac{}{\Gamma.A \vdash_{\mathrm{v}} A} \qquad \frac{t : \Gamma \vdash_{\mathrm{v}} A}{\Gamma.B \vdash_{\mathrm{v}} A}$$

$$\frac{\Gamma \vdash_{\mathrm{v}} A}{\Gamma \vdash_{\mathrm{ne}} A} \qquad \frac{\Gamma \vdash_{\mathrm{ne}} A \to B \quad \Gamma \vdash_{\mathrm{nf}} A}{\Gamma \vdash_{\mathrm{ne}} B}$$

$$\frac{\Gamma \vdash_{\mathrm{ne}} P}{\Gamma \vdash_{\mathrm{nf}} P} \qquad \frac{\Gamma.A \vdash_{\mathrm{nf}} B}{\Gamma \vdash_{\mathrm{nf}} A \to B}$$

Lemma : $\nvdash_{\mathrm{nf}} ((P \to Q) \to P) \to P$

Proof: Analyze possible derivations.

Normalisation theorem:

$$\frac{\Gamma \vdash A}{\Gamma \vdash_{\mathrm{nf}} A}$$ hence $\nvdash ((P \to Q) \to P) \to P$

- But how do we prove normalisation?

Kripke model

A Kripke model $K = (W, \leq, \Vdash)$ is given by

- A preordered set of worlds $(W, \leq)$.
- A monotone forcing relation $\Vdash \subseteq W \times X$: $\dfrac{w' \leq w \qquad w \Vdash P}{w' \Vdash P}$

Forcing

We recursively extend the forcing relation to:

propositions $w \Vdash A \to B = \forall w' \leq w . w' \Vdash A \to w' \Vdash B$

contexts $w \Vdash A_0 \ldots . A_n = w \Vdash A_0 \wedge \cdots \wedge w \Vdash A_n$

Lemma

Monotonicity holds for all propositions:

$$\frac{w' \leq w \qquad w \Vdash A}{w' \Vdash A}$$

Soundness

$$\frac{\Gamma \vdash A}{\forall w. w \Vdash \Gamma \rightarrow w \Vdash A}\ \text{sound}$$

$\nvdash ((P \to Q) \to P) \to P$ using a Kripke model

A countermodel

- $W = \{0, 1\}$ with $0 \leq 1$.
- $1 \Vdash P$

$$0 \nVdash ((P \to Q) \to P) \to P$$

hence using soundness

$$\nvdash ((P \to Q) \to P) \to P$$

How good are Kripke models ?

- We can refute some unprovable propositions using truthtables.
- We can refute more unprovable propositions using Kripke models.
- Are all unprovable propositions refutable by Kripke models?
- Or positively: are all propositions which hold in all Kripke models, provable.
- Even better there is one universal Kripke model U in which precisely the derivable propositions hold:

$$\frac{\forall w.w \Vdash \Gamma \rightarrow w \Vdash A}{\Gamma \vdash A}$$

Define: $\Gamma \vdash^* A_1 \ldots . A_n = \Gamma \vdash A_1 \wedge \ldots \Gamma \vdash A_n$, we can show:

1. $\Gamma \vdash^* \Gamma$
2. $$\frac{\Gamma \vdash^* \Delta \qquad \Delta \vdash A}{\Gamma \vdash A}$$
3. $$\frac{\Gamma \vdash^* \Delta \qquad \Delta \vdash^* \Theta}{\Gamma \vdash^* \Theta}$$

The universal model

$U = (\mathrm{Contexts}, \vdash^*, \vdash)$

- $(\mathrm{Contexts}, \vdash^*)$ is a preorder by 1,3
- $\vdash$ is monotone by 2

Completeness

quote and unquote

$$\frac{\Gamma \Vdash A}{\Gamma \vdash A}\mathit{quote} \qquad \frac{\Gamma \vdash A}{\Gamma \Vdash A}\mathit{unquote}$$

Proof: mutual induction over A.

Completeness

$$\frac{\forall \Delta . \Delta \Vdash^{*} \Gamma \rightarrow \Delta \Vdash A}{\Gamma \vdash A}\,\mathrm{Compl}$$

Proof: Combine quote and unquote.

Going back and forth

$$\cfrac{\cfrac{\Gamma \vdash A}{\forall \Delta . \Delta \Vdash^{*} \Gamma \rightarrow \Delta \Vdash A}\;\text{sound}}{\Gamma \vdash A}\;\text{compl}$$

- What have we achieved?
- We would like to obtain $\Gamma \vdash_{\mathrm{nf}} A$.
- Let's shrink the model...
- and revisit completeness.

1. $\Gamma \vdash_{\mathrm{v}}^{*} \Gamma$
2. $$\frac{\Gamma \vdash_{\mathrm{v}}^{*} \Delta \qquad \Delta \vdash_{x} A}{\Gamma \vdash_{x} A}$$
   with $x \in \{\mathrm{v}, \mathrm{ne}, \mathrm{nf}\}$
3. $$\frac{\Gamma \vdash_{\mathrm{v}}^{*} \Delta \qquad \Delta \vdash_{\mathrm{v}}^{*} \Theta}{\Gamma \vdash_{\mathrm{v}}^{*} \Theta}$$

The universal model (with normal forms)

$U = (\mathrm{Contexts}, \vdash_{\mathrm{v}}^{*}, \vdash_{\mathrm{ne}} (= \vdash_{\mathrm{nf}}))$

- $(\mathrm{Contexts}, \vdash_{\mathrm{v}}^{*})$ is a preorder by 1,3
- $\vdash_{\mathrm{ne}}$ is monotone by 2

Completeness (with normal forms)

quote and unquote

$$\frac{\Gamma \Vdash A}{\Gamma \vdash_{\mathrm{nf}} A}\mathit{quote} \qquad \frac{\Gamma \vdash_{\mathrm{ne}} A}{\Gamma \Vdash A}\mathit{unquote}$$

Completeness

$$\frac{\forall \Delta . \Delta \Vdash^{*} \Gamma \rightarrow \Delta \Vdash A}{\Gamma \vdash_{\mathrm{nf}} A}\mathrm{Compl}$$

Proof: Combine quote and unquote.

Normalisation from completeness

$$\dfrac{\dfrac{\Gamma \vdash A}{\forall \Delta . \Delta \Vdash^* \Gamma \rightarrow \Delta \Vdash A}\,\text{sound}}{\Gamma \vdash_{\text{nf}} A}\,\text{compl}$$

- Normalisation is a consequence of completeness!
- We adjust the model and check the proof to show that completeness always produces normal forms.
- Once we have normalisation we don't need the models anymore!

From NBC to NBE

NBC	NBE
mininal logic	λ-calculus (CCC)
preorder	category
monotone	functorial
Kripke model	presheaf model
soundness	presheaves are cartesian closed

Adding connectives

Conjunction

$$w \Vdash A \wedge B = w \Vdash A \wedge w \Vdash B$$
$$w \Vdash \top = \top$$

Soundness ok
Completeness ok

Disjunction

$$w \Vdash A \vee B = w \Vdash A \vee w \Vdash B$$
$$w \Vdash \bot = \bot$$

Soundness ok
Completeness ???

The problem with disjunction

$$U = (\text{Contexts}, \vdash^*, \vdash)$$

$$\dfrac{\dfrac{P \vee Q \vdash P \vee Q}{P \vee Q \Vdash P \vee Q}\,\text{unquote}}{\dfrac{(P \vee Q \Vdash P) \vee (P \vee Q \Vdash Q)}{(P \vee Q \vdash P) \vee (P \vee Q \vdash Q)}\,\text{quote}}$$

- But aren't Kripke models complete for intuitionistic logic?
- Yes, but the universal model has to be constructed differently.
- Contexts are replaced by *saturated contexts*...
- The construction of the universal model now requires decidability:

$$\Gamma \vdash A \vee \Gamma \nvdash A$$

- Indeed, completeness for Kripke models for intuitionistic predicate logic is **not** provable intuitionistically.
- Instead, we will consider a different class of models.

Beth model

A Beth model $B = (W, \leq, \Vdash, \lhd)$ is given by

- A Kripke model $(W, \leq, \Vdash)$.
- A covering relation $\lhd \subseteq W \times \mathcal{P} W$ such that:

trivial $w \lhd \{w' \mid w' \leq w\}$

monotone $$\frac{w \lhd P \quad w' \leq w}{w' \lhd P}$$

union $$\frac{w \leq P \qquad \forall w' \in P. w' \lhd Q}{w \lhd Q}$$

paste $$\frac{w \lhd P \quad \forall w' \in P. w' \Vdash Q}{w \Vdash Q}$$

Forcing

We extend the forcing relation:

$$w \Vdash A \vee B = \exists P. w \lhd P \wedge \forall w' \in P. w' \lhd A \vee w' \lhd B$$
$$w \Vdash \bot = w \lhd \{\}$$

Soundness for Beth models

Lemma:

Monotonicity and paste hold for all formulas:

$$\frac{w \lhd A \quad \forall w' \in P.w' \Vdash A}{w \Vdash A} \qquad \frac{w' \leq w \qquad w \Vdash A}{w' \Vdash A}$$

Soundness:

$$\frac{\Gamma \vdash A}{\forall w.w \Vdash \Gamma \rightarrow w \Vdash A}\,\text{sound}$$

The universal Beth model

$U = (\mathrm{Contexts}, \vdash^*, \vdash, \lhd)$

- $(\mathrm{Contexts}, \vdash^*, \vdash)$ is the universal Kripke model.
- $\lhd$ is defined inductively:
  1. $\Gamma \lhd \{\Delta \mid \Delta \leq \Gamma\}$
  2. $$\frac{\Gamma \vdash \bot}{\Gamma \lhd P}$$
  3. $$\frac{\Gamma \vdash A \vee B \quad \Gamma.A \lhd P \quad \Gamma.B \lhd Q}{\Gamma \lhd P \cup Q}$$

Completeness

$$\frac{\forall \Delta.\Delta \Vdash^* \Gamma \to \Delta \Vdash A}{\Gamma \vdash A}\,\mathrm{Compl}$$

Proof: Extend quote and unquote.

Normalisation?

- Left as an exercise.
- First step: come up with a good notion of normal form...

From NBC to NBE (contd)

NBC	NBE
$\lhd$	Grothendieck topology
Beth model	sheaf model

Summary

- We have solved simpler problems: the existence of normal forms.
- We have ignored equality of derivations.
- We have shown that normalisation can be obtained by a modified universal model.
- NBE can be recovered by moving to corresponding proof-relevant constructions.
- Now for something completely different:
  Why is it hard to formalize Type Theory in Type Theory?