Functional Quantum Programming

Thorsten Altenkirch

University of Nottingham

based on joint work with Jonathan Grattage

and discussions with V.P. Belavkin

Background

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Assumption: Nature is fair. . .

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- Richard Josza, QPL 2004: *We need to develop quantum thinking!*

QML

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  FCC Finite classical computations

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- Compiler under construction (Jonathan)

Example: Hadamard operation

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Matrix

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QML

$$\begin{aligned} \mathsf{H}\, x : \mathcal{Q}_2 = \mathtt{if}^{\circ}\ x\ & \mathtt{then}\ \{\mathtt{qfalse} \mid (-1)\mathtt{qtrue}\} \\ & \mathtt{else}\ \{\mathtt{qfalse} \mid \mathtt{qtrue}\} \end{aligned}$$

Related Work

- P. Zuliani, 2001, *Quantum Programming*
- S. Abramsky and B. Coecke, 2004, *A Categorical Semantics of Quantum Protocols*
- S-C. Mu and R. S. Bird, 2001, *Quantum functional programming*
- A. Sabry, 2003, *Modeling quantum computing in Haskell*
- J. Karczmarczuk, 2003, *Structure and interpretation of quantum mechanics: a functional framework*
- P. Selinger, 2002, *Towards a Quantum Programming Language*
- A. van Tonder, 2003, *A Lambda Calculus for Quantum Computation*

Something we know well . . .

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- Quantum mechanics is time-reversible. . .
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- **However:** Newtonian mechanics, Maxwellian electrodynamics is also time-reversible. . .
- . . . hence classical computation **should be** based on reversible operations.

Classical computations (FCC)

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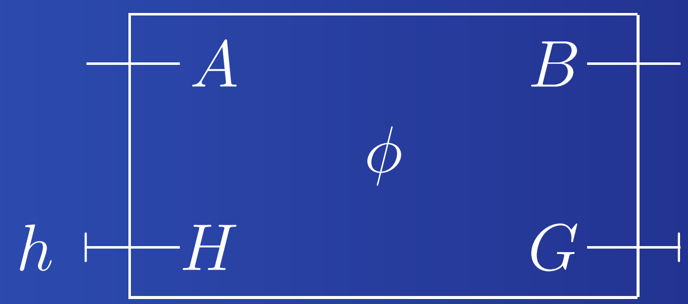
Given finite sets A (input) and B (output):



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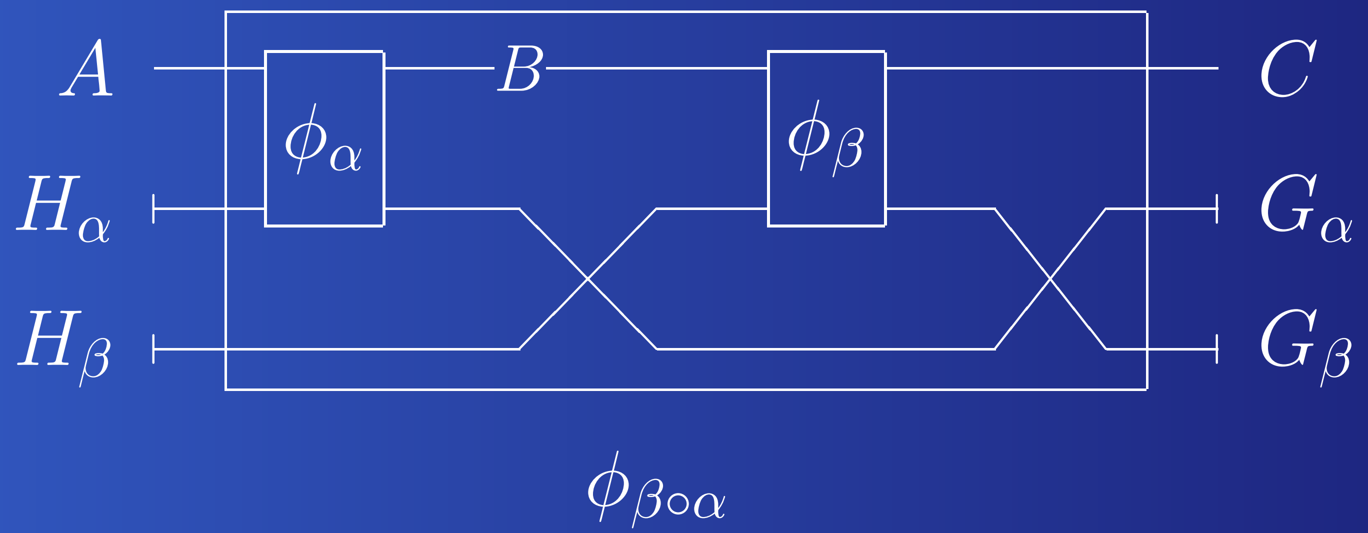
Given finite sets A (input) and B (output):

- a finite set of initial heaps H,
- an initial heap $h \in H$,
- a finite set of garbage states G,
- a bijection $\phi \in A \times H \simeq B \times G$,

Composing classical computations

Composing classical computations



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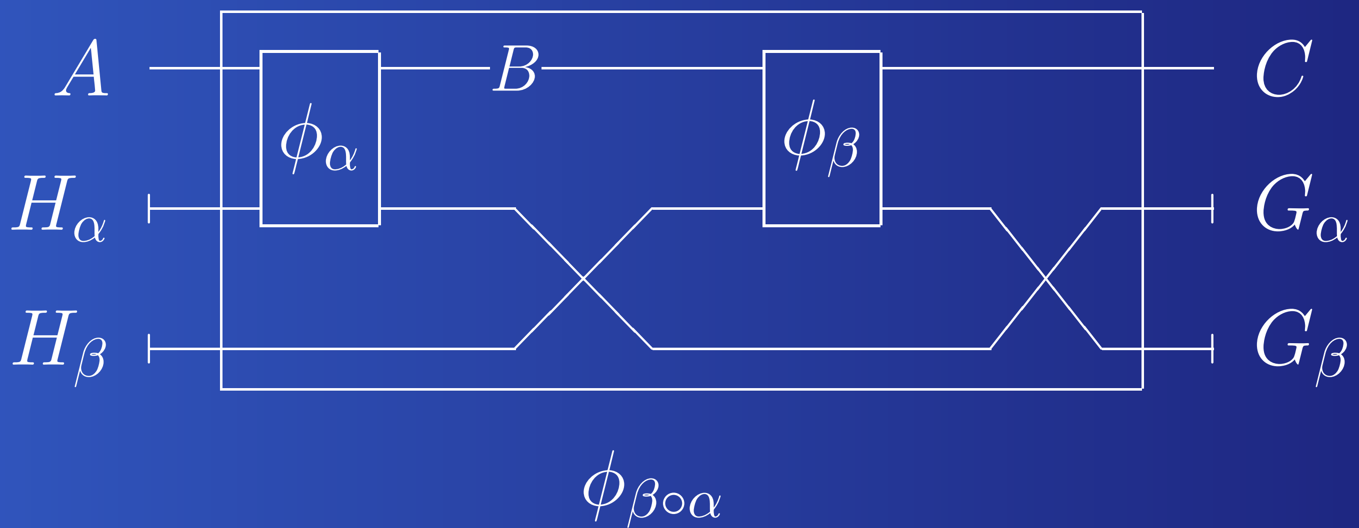


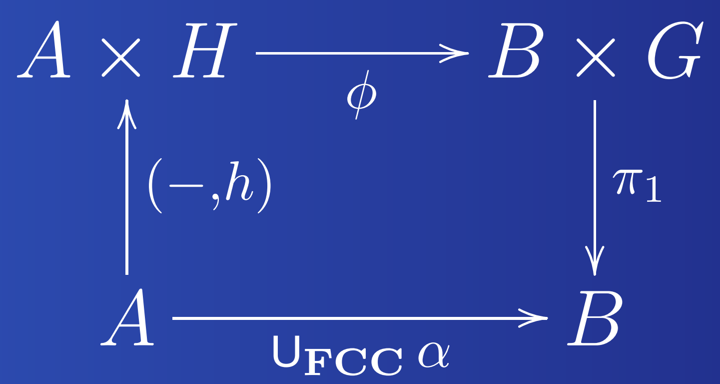
Exercise: Define I.

Extensional equality

Extensional equality

Every computation α gives rise to a function

$\mathsf{U}_{\mathbf{FCC}}\,\alpha \in A \to B$



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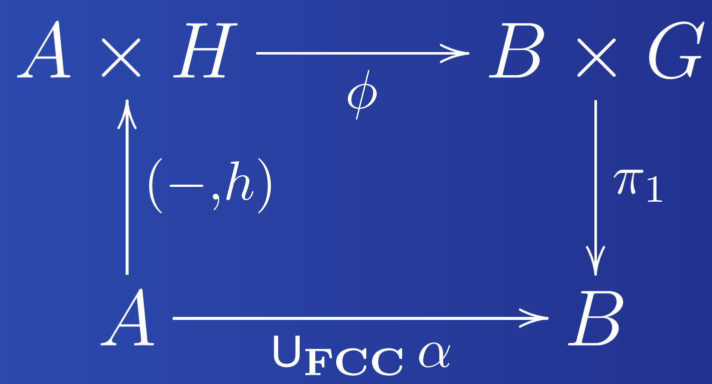

$$\alpha =_{\mathsf{ext}} \beta, \text{ if } \mathsf{U}_{\mathbf{FCC}}\,\alpha = \mathsf{U}_{\mathbf{FCC}}\,\beta$$

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$$
\begin{array}{ccc}
A \times H & \xrightarrow[\phi]{} & B \times G \\
\big\uparrow{\scriptstyle (-,h)} & & \big\downarrow{\scriptstyle \pi_1} \\
A & \xrightarrow[\mathsf{U}_{\mathbf{FCC}}\,\alpha]{} & B
\end{array}
$$

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$\mathbf{FCC}$:

Objects fi nite sets

Morphisms computations $/ =_{\mathsf{ext}}$.

$\mathsf{U}_{\mathbf{FCC}}$

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- **Exercise:** $\mathsf{U}_{\mathbf{FCC}}$ is full!

Coming next: Quantum computations

Develop FQC analogously to FCC. . .

Linear algebra revision

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$\mathbb{C}\,A = A \to \mathbb{C}$ is a **Hilbert space**.

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Norm of a vector:

$\|v\| = \Sigma_{a \in A}(v\,a)^*(v\,a) \in \mathbb{R}^+$,

Unitary operators:

A unitary operator $\phi \in A \multimap_{\text{unitary}} B$ is a linear isomorphism that preserves the norm.

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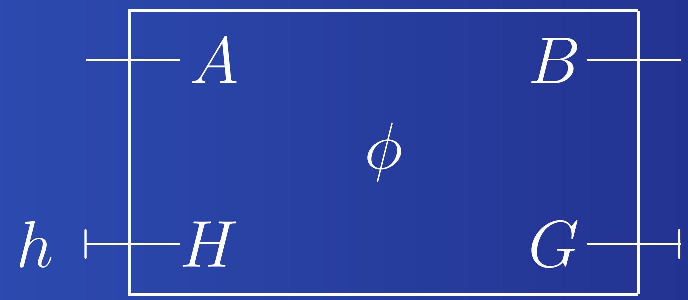

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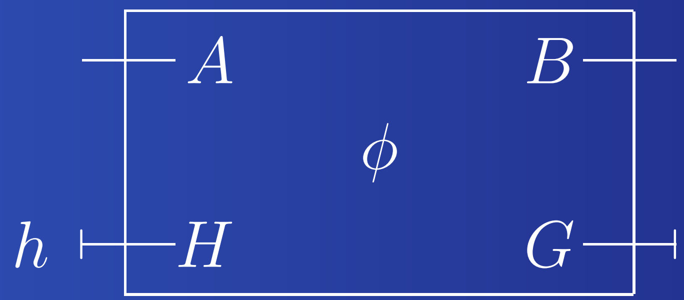

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- a unitary operator $\phi \in A \otimes H \multimap_{\text{unitary}} B \otimes G$.

Composing quantum computations

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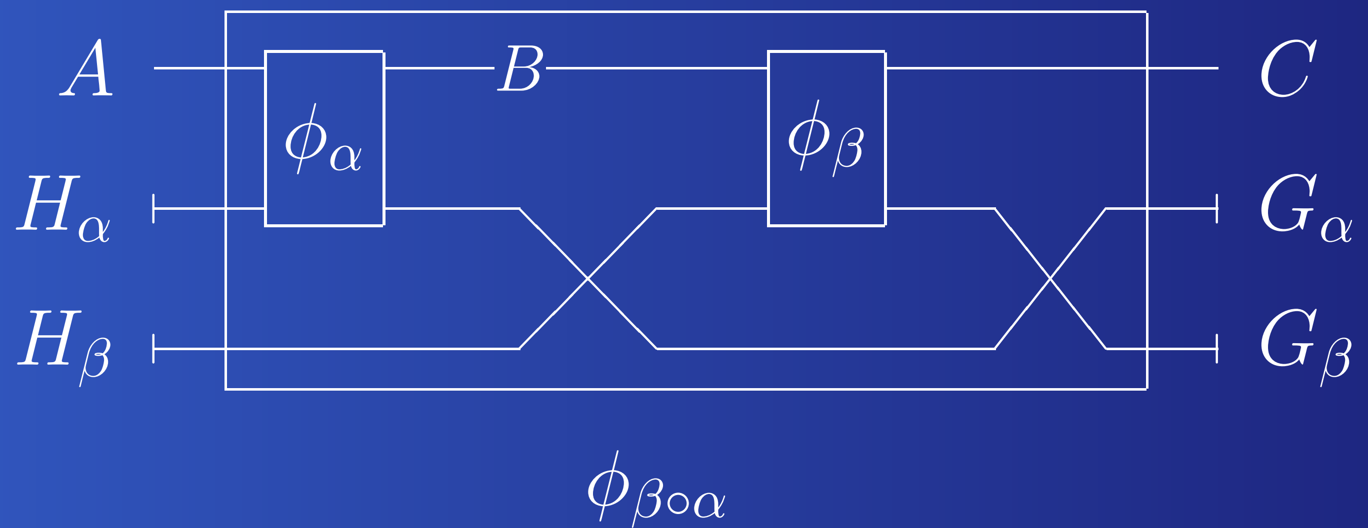

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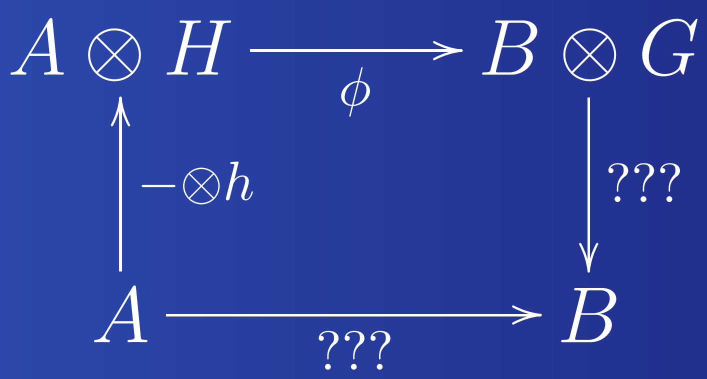

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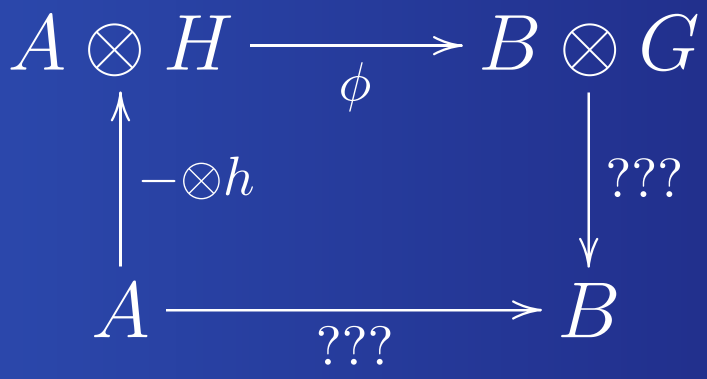

- **Indeed:** Forgetting part of a **pure state** results in a **mixed state**.

Density Operators

A mixed state on A is given by a **density operator**

$$\rho \in A \multimap A$$

such that all eigenvalues are positive reals

$$\hat{\rho}\, v = \lambda v \implies \lambda \in \mathbb{R}^{+}$$

and has a unit trace

$$\Sigma a \in A . v\, a = 1$$

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- Partial trace:

$$\mathrm{tr}_{A,G} \in A \otimes G \multimap_{\text{super}} A$$

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Every computation α gives rise to a superoperator $\mathsf{U}\,\alpha \in A \multimap_{\mathsf{super}} B$

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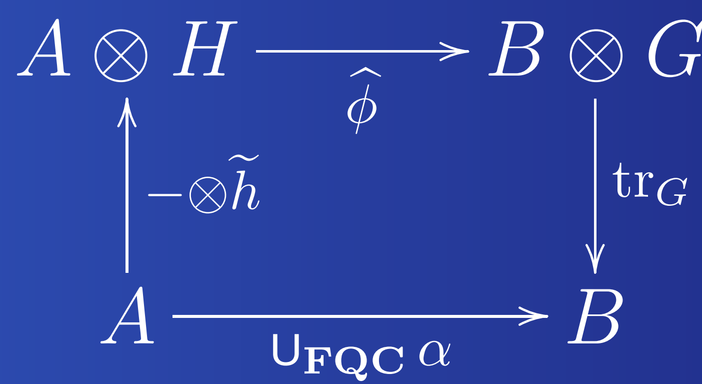

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FCC:
- **Objects** finite sets
- **Morphisms** computations / $=_{\text{ext}}$.

UFQC

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- $\mathsf{U}_{\mathbf{FQC}}$ is full!

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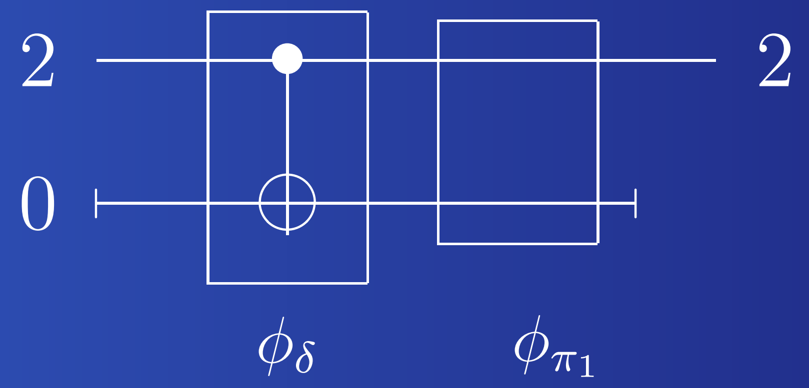
Classical vs quantum

classical	quantum
finite sets	finite dimensional Hilbert spaces
bijections	unitary operators
cartesian product ($\times$)	tensor product ($\otimes$)
functions	superoperators
projections	partial trace

Decoherence

Decoherence



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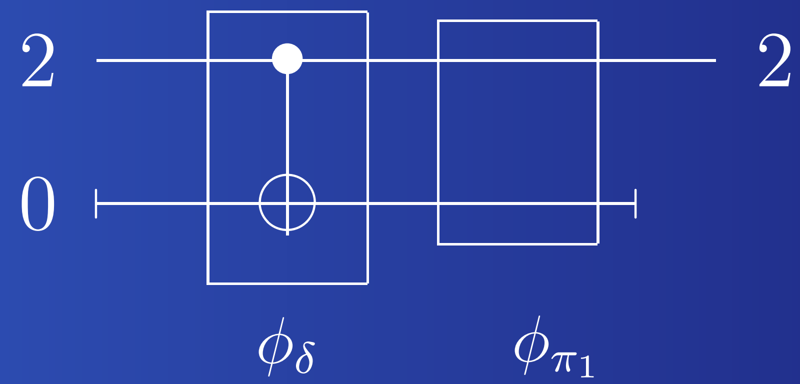
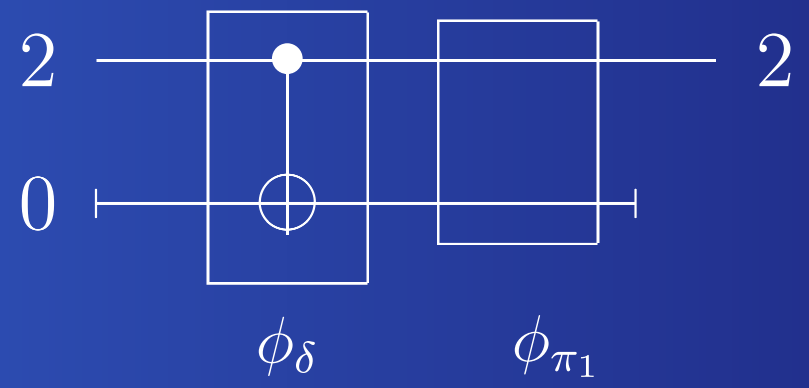


Classically

$$\pi_1 \circ \delta = \mathrm{I}$$

Decoherence



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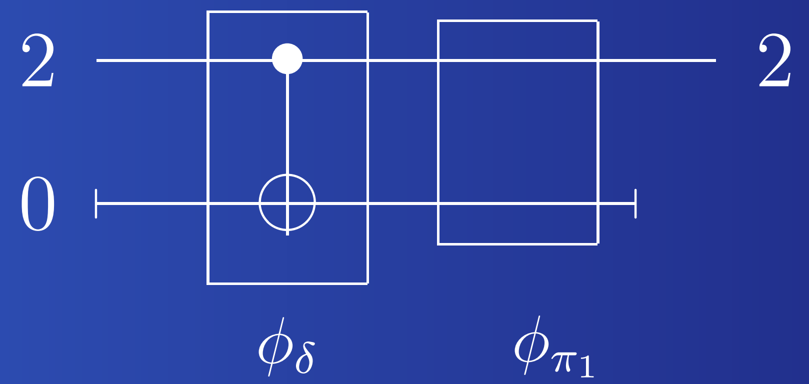

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Quantum

input: $\{\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|0\rangle\}$

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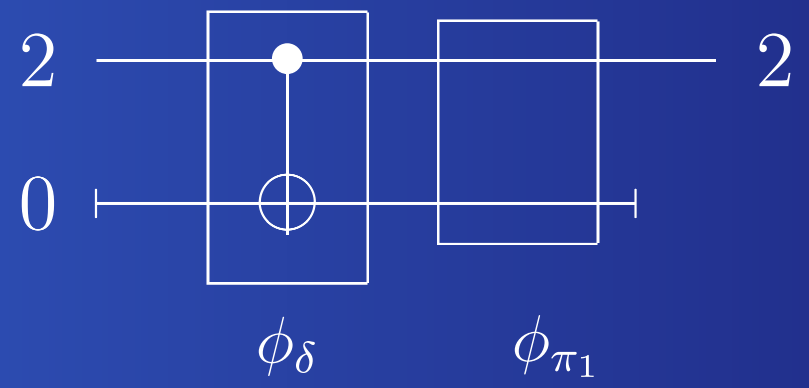

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Quantum

input: $\{ \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |0\rangle \}$

output: $\frac{1}{2}\{|0\rangle\} + \frac{1}{2}\{|1\rangle\}$

QML basics

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  no weakening but contraction.

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- QML is based on strict linear logic
  no weakening but contraction.
- QML types: $1, \sigma \otimes \tau, \sigma \oplus \tau$

Interpretation of types

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$$
\begin{aligned}
|1| &= 0 \\
|\sigma \sqcup \tau| &= \max\{|\sigma|, |\tau|\} \\
|\sigma \oplus \tau| &= |\sigma \sqcup \tau| + 1 \\
|\sigma \otimes \tau| &= |\sigma| + |\tau|
\end{aligned}
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Interpretation of types

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$$[\![\sigma]\!] = 2^{|\sigma|}$$

⊗ on contexts

$\otimes$ on contexts

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\Gamma, x:\sigma \otimes \Delta, x:\sigma &= (\Gamma \otimes \Delta), x:\sigma \\
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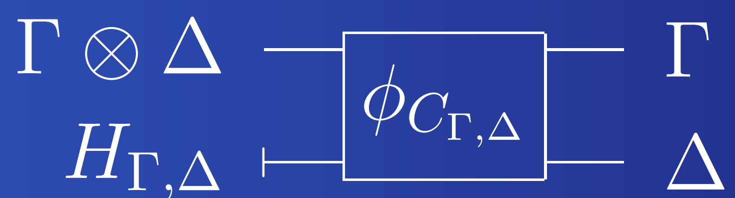
\bullet \otimes \Delta &= \Delta
\end{aligned}
$$



The let-rule

The let-rule

$$\frac{\Gamma \vdash t : \sigma \qquad \Delta, x : \sigma \vdash u : \tau}{\Gamma \otimes \Delta \vdash \texttt{let}\ x = t\ \texttt{in}\ u : \tau}\ \text{let}$$

The let-rule

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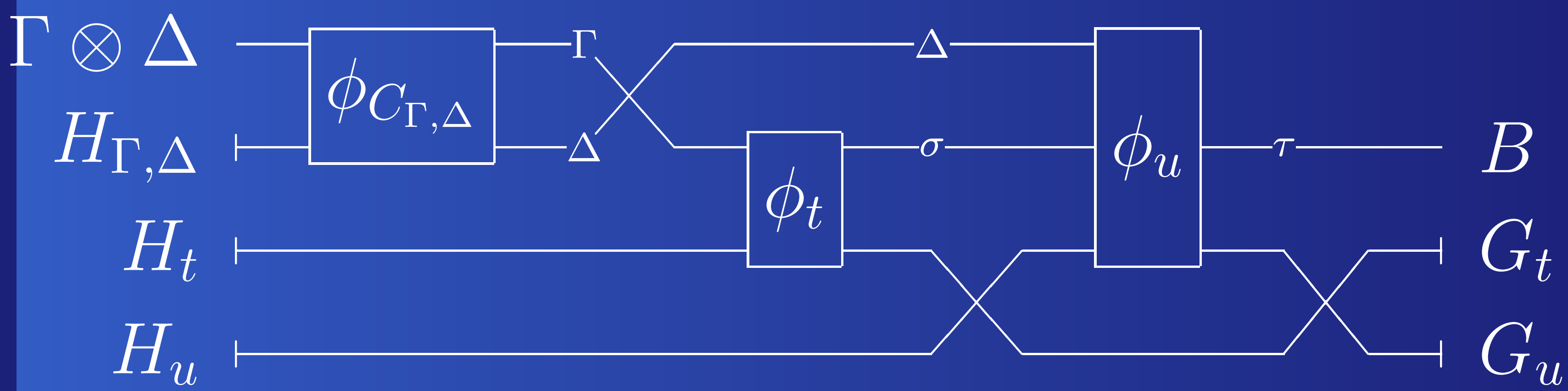

The var-rule

The var-rule

$$\frac{}{\Gamma, x : \sigma \vdash x^{\mathsf{dom}\,\Gamma} : \sigma}\ \text{var}$$

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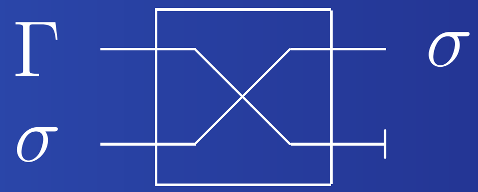
$$\frac{}{\Gamma, x : \sigma \vdash x^{\mathsf{dom}\,\Gamma} : \sigma}\ \mathrm{var}$$



Example

$$y : \mathcal{Q}_2 \vdash \texttt{let}\ x = y\ \texttt{in}\ x^{\{\}} : \mathcal{Q}_2$$

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$\otimes$-intro

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$$\frac{\Gamma \vdash t : \sigma \quad \Delta \vdash u : \tau}{\Gamma \otimes \Delta \vdash (t, u) : \sigma \otimes \tau} \otimes \mathrm{intro}$$

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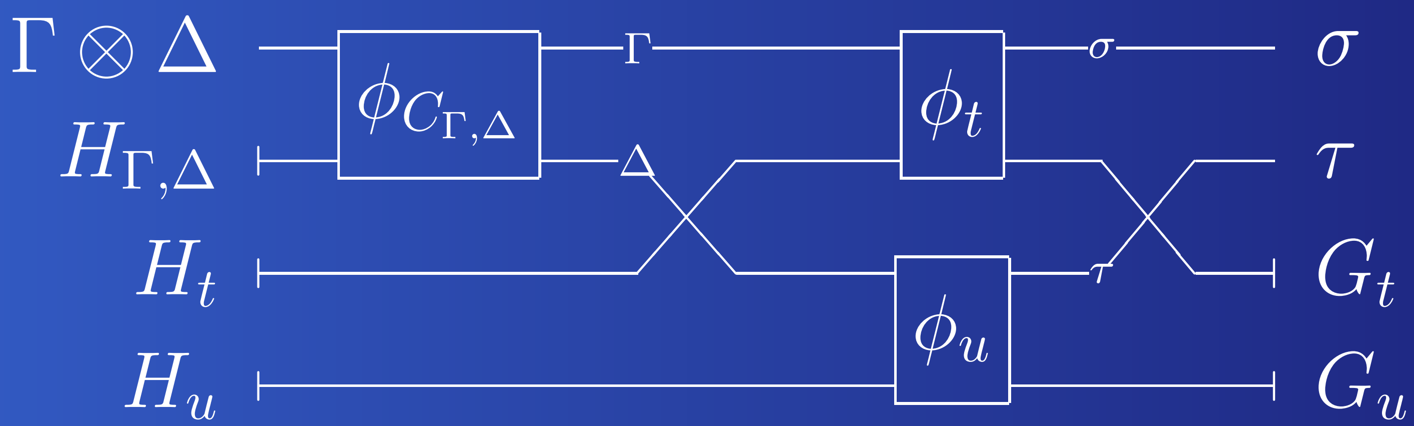


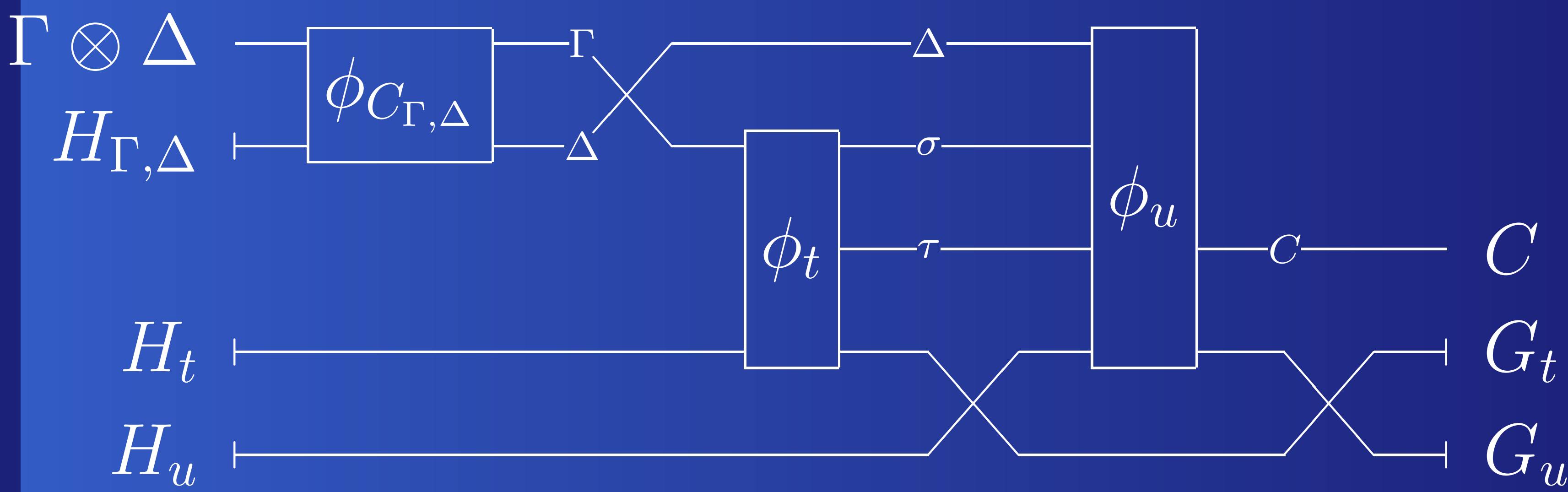
$\otimes$-elim

$\otimes$-elim

$$\frac{\Gamma \vdash t : \sigma \otimes \tau \qquad \Delta, x : \sigma, y : \tau \vdash u : C}{\Gamma \otimes \Delta \vdash \texttt{let}\ (x, y) = t\ \texttt{in}\ u : C} \otimes \mathrm{elim}$$

⊗-elim

$$\frac{\Gamma \vdash t : \sigma \otimes \tau \qquad \Delta, x : \sigma, y : \tau \vdash u : C}{\Gamma \otimes \Delta \vdash \texttt{let}\ (x, y) = t\ \texttt{in}\ u : C} \otimes \text{elim}$$



Example

$$p : \mathcal{Q}_2 \otimes \mathcal{Q}_2 \vdash \texttt{let}\ (x, y) = p\ \texttt{in}\ (y^{\{\}}, x^{\{\}}) : \mathcal{Q}_2 \otimes \mathcal{Q}_2$$

Example

$$p : \mathcal{Q}_2 \otimes \mathcal{Q}_2 \vdash \texttt{let}\ (x, y) = p\ \texttt{in}\ (y^{\{\}}, x^{\{\}}) : \mathcal{Q}_2 \otimes \mathcal{Q}_2$$

$$p : \mathcal{Q}_2 \otimes \mathcal{Q}_2 \vdash \texttt{let}\ (x, y) = p\ \texttt{in}\ (y^{\{p\}}, x^{\{p\}}) : \mathcal{Q}_2 \otimes \mathcal{Q}_2$$

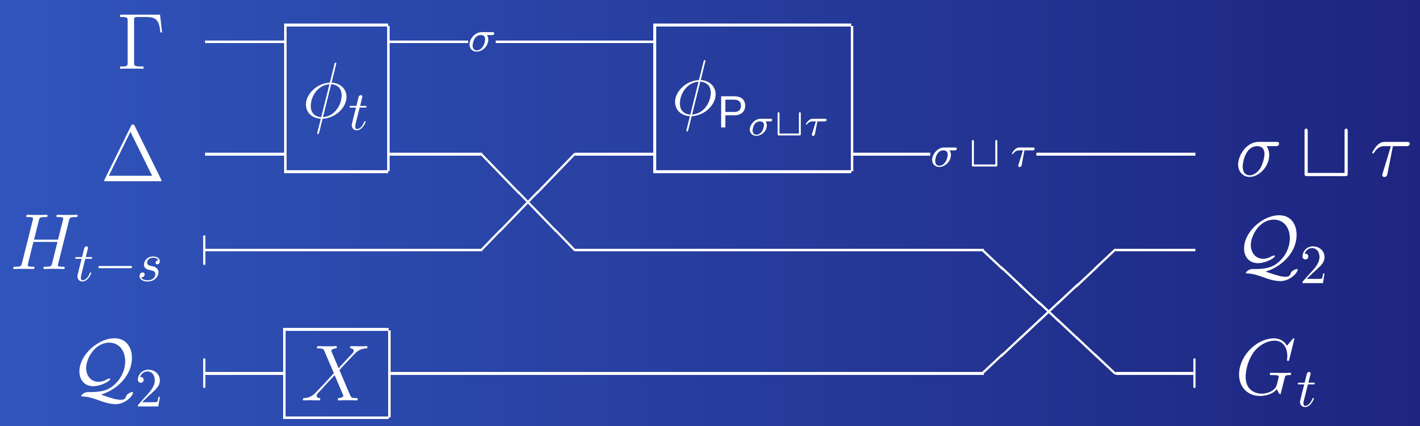
⊕-intro

$\oplus$-intro

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathtt{inl}\, t : A \oplus B}$$

$\oplus$-intro

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \texttt{inl}\, t : A \oplus B}$$



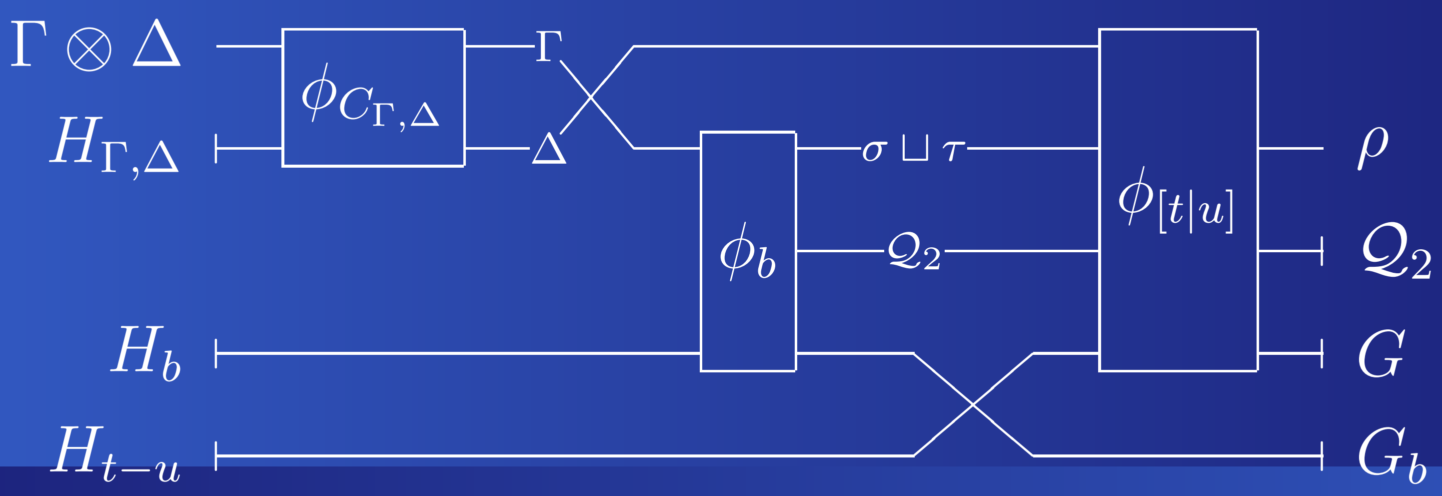
$\oplus$-elim

$\oplus$-elim

$$\frac{\Gamma \vdash c : \sigma \oplus \tau \qquad \Delta, x : \sigma \vdash t : \rho \qquad \Delta, y : \tau \vdash u : \rho}{\Gamma \otimes \Delta \vdash \texttt{case}\ c\ \texttt{of}\ \{\texttt{inl}\ x \Rightarrow t \mid \texttt{inr}\ y \Rightarrow u\} : \rho}\ +\mathrm{elim}$$

$\oplus$-elim

$$\frac{\Gamma \vdash c : \sigma \oplus \tau \qquad \Delta, x : \sigma \vdash t : \rho \qquad \Delta, y : \tau \vdash u : \rho}{\Gamma \otimes \Delta \vdash \texttt{case}\ c\ \texttt{of}\ \{\texttt{inl}\, x \Rightarrow t \mid \texttt{inr}\, y \Rightarrow u\} : \rho} + \mathrm{elim}$$



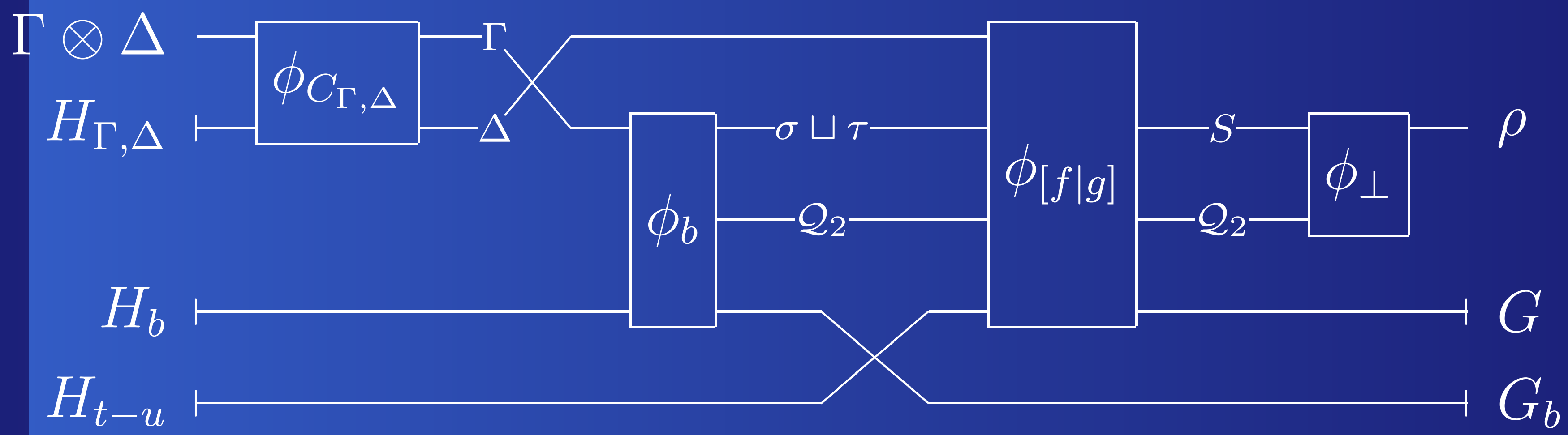
⊕-elim decoherence-free

$\oplus$-elim decoherence-free

$$\frac{\Gamma \vdash c : \sigma \oplus \tau \qquad \Delta, x : \sigma \vdash t : \rho \qquad \Delta, y : \tau \vdash u : \rho, \quad t \perp u}{\Gamma \otimes \Delta \vdash \texttt{case}^{\circ}\ b\ \texttt{of}\ \{\texttt{inl}\, x \Rightarrow t \,|\, \texttt{inr}\, y \Rightarrow u\} : \rho} + \mathrm{elim}^{\circ}$$

$\oplus$-elim decoherence-free

$$\frac{\Gamma \vdash c : \sigma \oplus \tau \qquad \Delta, x : \sigma \vdash t : \rho \qquad \Delta, y : \tau \vdash u : \rho, \quad t \perp u}{\Gamma \otimes \Delta \vdash \texttt{case}^{\circ}\ b\ \texttt{of}\ \{\texttt{inl}\ x \Rightarrow t \mid \texttt{inr}\ y \Rightarrow u\} : \rho} + \mathrm{elim}^{\circ}$$



Orthogonality

$$\frac{}{\texttt{inl}\ t \perp \texttt{inr}\ u} \qquad \frac{t \perp u}{\texttt{inl}\ t \perp \texttt{inl}\ u \quad \texttt{inr}\ t \perp \texttt{inr}\ u}$$

$$\frac{t \perp u}{(t, v) \perp (u, w) \quad (v, t) \perp (w, u)}$$

Semantics of $\perp$

$$[\![t \perp u]\!] = (S, \phi, f, g)$$

- S finite set.
- $\phi \in \mathcal{Q}_2 \otimes S \multimap_{\text{unitary}} [\![\sigma]\!]$
- $f \in \mathbf{FQC}\,[\![\Gamma]\!]\,S$
  $g \in \mathbf{FQC}\,[\![\Gamma]\!]\,S$
- $[\![t]\!] = \phi \circ (\mathrm{true} \otimes -) \circ f,$
  $[\![u]\!] = \phi \circ (\mathrm{false} \otimes -) \circ g$

Superpositions

$$\frac{\Gamma \vdash t, u : \sigma \qquad t \perp u \qquad ||\lambda||^2 + ||\lambda'||^2 = 1 \quad \lambda, \lambda' \neq 0}{\Gamma \vdash \{(\lambda)t \,|\, (\lambda')u\} : \sigma}$$

$$\equiv \ \texttt{if}^\circ \ \{(\lambda)\texttt{qtrue} \,|\, (\lambda')\texttt{qfalse}\} \ \texttt{then} \ t \ \texttt{else} \ u$$

Example: Deutsch's algorithm

$$
\begin{aligned}
\mathrm{Eq}\, a : \mathcal{Q}_2, b : \mathcal{Q}_2 = \texttt{let}\,(x,y) = \;& \texttt{if}^\circ\ \{\texttt{qfalse} \mid (-1)\texttt{qtrue}\} \\
& \texttt{then}\ \ (\texttt{qtrue},\ \texttt{if}\ \ a \\
& \qquad\qquad\qquad \texttt{then}\ \{\texttt{qfalse} \mid (-1)\texttt{qtrue}\} \\
& \qquad\qquad\qquad \texttt{else}\ \{\texttt{qfalse} \mid \texttt{qtrue}\}) \\
& \texttt{else}\ \ (\texttt{qfalse},\texttt{if}\ \ b \\
& \qquad\qquad\qquad \texttt{then}\ \{\texttt{qfalse} \mid (-1)\texttt{qtrue}\} \\
& \qquad\qquad\qquad \texttt{else}\ \{\texttt{qfalse} \mid \texttt{qtrue}\}) \\
\texttt{in}\ x & \\
: \ \mathcal{Q}_2 &
\end{aligned}
$$

Future work

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- Higher order

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- High level reasoning principles for QML programs

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- Infi nite or indexed?