The University of Nottingham

An Algebra of Pure Quantum Programming

based on joint work with:

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University of Nottingham

QML

QML

- Functional core language for quantum programming

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- Finitary, first order

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- Denotational semantics:
  superoperators on finite dimensional spaces

Main design ideas

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- Contraction by sharing

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$$\delta\ x = (x, x)$$

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$$\delta\ (\mathit{false} +_Q \mathit{true}) \equiv (\mathit{false}, \mathit{false}) +_Q (\mathit{true}, \mathit{true})$$

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- Reversible **if**$^\circ$ and irreversible **if**.

Reversible $\mathbf{if}^\circ$ and irreversible $\mathbf{if}$

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$$\neg^c \in Q_2 \multimap Q_2$$

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$$cswap \in Q_2 \multimap Q_2 \otimes Q_2 \multimap Q_2 \otimes Q_2$$
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$$cswap' \in Q_2 \multimap Q_2 \otimes Q_2 \multimap Q_2 \otimes Q_2$$

$$cswap'\ x\ (y, z) = \mathbf{if}\ x\ \mathbf{then}\ (z, y)\ \mathbf{else}\ (y, z)$$

is well-typed, since **if** does not require orthogonality.

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  Denotational semantics: sets and (injective) functions
- Sound and complete.
- Completeness also gives rise to a normalisation algorithm
  (*Normalisation by evaluation*).

Example

$$H \in Q_2 \multimap Q_2$$

$$\begin{aligned} H\ x = \mathbf{if}^{\circ}\ x \\ &\mathbf{then}\ (\mathit{false} + (-1) * \mathit{true}) \\ &\mathbf{else}\ \ (\mathit{false} + \mathit{true}) \end{aligned}$$

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$$\vdash H\ (H\ x) \equiv x$$

Derivation

$$
\begin{aligned}
H\ (H\ x) &= \mathbf{if}^\circ\ (\mathbf{if}^\circ\ x \\
&\qquad\qquad \mathbf{then}\ (\mathit{false} + (-1) * \mathit{true}) \\
&\qquad\qquad \mathbf{else}\ \ (\mathit{false} + \mathit{true})) \\
&\quad \mathbf{then}\ (\mathit{false} + (-1) * \mathit{true}) \\
&\quad \mathbf{else}\ \ (\mathit{false} + \mathit{true})
\end{aligned}
$$

-- by commuting conversion for $\mathbf{if}^\circ$

$$
\begin{aligned}
&= \mathbf{if}^\circ\ x \\
&\quad \mathbf{then}\ \mathbf{if}^\circ\ (\mathit{false} + (-1) * \mathit{true}) \\
&\qquad\qquad \mathbf{then}\ (\mathit{false} + (-1) * \mathit{true}) \\
&\qquad\qquad \mathbf{else}\ \ (\mathit{false} + \mathit{true}) \\
&\quad \mathbf{else}\ \ \mathbf{if}^\circ\ (\mathit{false} + \mathit{true}) \\
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&\quad \mathbf{else}\ \ \mathbf{if}^{\circ}\ (\mathit{false} + \mathit{true}) \\
&\qquad\qquad \mathbf{then}\ (\mathit{false} + (-1) * \mathit{true}) \\
&\qquad\qquad \mathbf{else}\ \ (\mathit{false} + \mathit{true})
\end{aligned}
$$

-- by $\mathbf{if}^{\circ}$

$$
\begin{aligned}
&= \mathbf{if}^{\circ}\ x \\
&\quad \mathbf{then}\ (\mathit{false} - \mathit{false} + \mathit{true} + \mathit{true}) \\
&\quad \mathbf{else}\ \ (\mathit{false} + \mathit{false} + \mathit{true} - \mathit{true})
\end{aligned}
$$

Derivation

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& \textbf{if}^\circ\ x \\
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& \textbf{else}\ \ (\mathit{false} + \mathit{false} + \mathit{true} - \mathit{true})
\end{aligned}$$

-- by simplification and normalisation

$$= \textbf{if}^\circ\ x\ \textbf{then}\ \mathit{true}\ \textbf{else}\ \mathit{false}$$

-- by η-rule for $\textbf{if}^\circ$

$$= x$$

Classical vs. quantum semantics

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Classical $$\frac{\Gamma \vdash^{C} t : \sigma}{[\![t]\!]^{C} \in [\![\Gamma]\!] \to [\![\sigma]\!]}$$

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Quantum $$\frac{\Gamma \vdash^{\mathrm{Q}} t : \sigma}{[\![t]\!]^{\mathrm{Q}} \in \mathbf{Q}^{\circ}\, [\![\Gamma]\!]\, [\![\sigma]\!]}$$

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where $\mathbf{Q}^{\circ}\, A\, B$ is the set of linear, isometric ($\langle v|w\rangle = \langle f\, v|f\, w\rangle$) functions on the spaces with base A and B (finite).

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Classical equations for $\mathbf{if}^\circ$

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β

$$\begin{aligned} \mathbf{if}^\circ\ \mathit{false}\ \mathbf{then}\ t\ \mathbf{else}\ u &\equiv u \\ \mathbf{if}^\circ\ \mathit{true}\ \mathbf{then}\ t\ \mathbf{else}\ u &\equiv t \end{aligned}$$

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η

$$\mathbf{if}^\circ\ t\ \mathbf{then}\ \mathit{true}\ \mathbf{else}\ \mathit{false} \equiv t$$

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η

$$\mathbf{if}^\circ\ t\ \mathbf{then}\ \mathit{true}\ \mathbf{else}\ \mathit{false} \equiv t$$

Commuting conversion

$$\begin{array}{ll} \mathbf{let}\ p = \mathbf{if}^\circ\ t & \equiv\ \ \mathbf{if}^\circ\ t \\ \quad \mathbf{then}\ u_0 & \quad \mathbf{then}\ \mathbf{let}\ p = u_0\ \mathbf{in}\ e \\ \quad \mathbf{else}\ u_1 & \quad \mathbf{else}\ \ \mathbf{let}\ p = u_1\ \mathbf{in}\ e \\ \mathbf{in}\ e & \end{array}$$

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$$\mathbf{let}\ p = \mathit{val}\ \mathbf{in}\ u \equiv u\,[\mathit{val}\ /\ p]$$

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$$\mathbf{let}\ x = t\ \mathbf{in}\ x \equiv t$$

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$$\mathbf{let}\ p = t\ \mathbf{in}\ \mathbf{let}\ q = u\ \mathbf{in}\ e \equiv \mathbf{let}\ q = u\ \mathbf{in}\ \mathbf{let}\ p = t\ \mathbf{in}\ e$$

quote (closed)

$$q^{\sigma} \in [\![\sigma]\!] \to \mathrm{Val}^{\mathrm{C}}\sigma$$

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Lemma (Adequacy)
The equation $\vdash^{\mathrm{C}} q^\sigma([\![\vdash t : \sigma]\!]^{\mathrm{C}}) \equiv t : \sigma$ is derivable.

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We show

$$\frac{\gamma \in [\![\Gamma]\!]^{\mathrm{C}} \quad u = q^\Gamma \gamma}{q^\sigma([\![t]\!]^{\mathrm{C}} \gamma) \equiv \mathbf{let}^* \ \Gamma = u \ \mathbf{in} \ t}$$

by induction over derivations.

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$$\begin{array}{rcl}
\phi_{\Gamma,x:\mathcal{Q}_2} & \in & (\mathrm{Tm}\,(\Gamma, x : \mathcal{Q}_2)\,\sigma) \to (\mathrm{Tm}\,\Gamma\,\sigma) \times (\mathrm{Tm}\,\Gamma\,\sigma) \\
\phi_{x:\mathcal{Q}_2}\,t & = & (\texttt{let}\ x = \mathrm{false}\ \texttt{in}\ t, \texttt{let}\ x = \mathrm{true}\ \texttt{in}\ t) \\
\phi^{-1}_{\Gamma,x:\mathcal{Q}_2}(t,u) & = & \texttt{if}^{\circ}\ x\ \texttt{then}\ t\ \texttt{else}\ u
\end{array}$$

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$$\begin{array}{rcl} q^{\sigma}_{\bullet}(f) & = & q^{\sigma}\ f \\ q^{\sigma}_{\Gamma,x:\mathcal{Q}_2}(f) & = & \phi^{-1}_{\Gamma,x:\mathcal{Q}_2} \circ (q^{\sigma}_{\Gamma} \times q^{\sigma}_{\Gamma}) \circ \llbracket \phi_{\Gamma,x:\mathcal{Q}_2} \rrbracket \end{array}$$

Normalisation and completeness

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Proposition: Completeness

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Quantum equations

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($\mathbf{if}^\circ$)

$$\mathbf{if}^\circ\ (t_0 + t_1)\ \mathbf{then}\ u_0\ \mathbf{else}\ u_1$$
$$\equiv (\mathbf{if}^\circ\ t_0\ \mathbf{then}\ u_0\ \mathbf{else}\ u_1) + (\mathbf{if}^\circ\ t_1\ \mathbf{then}\ u_0\ \mathbf{else}\ u_1)$$

$$\mathbf{if}^\circ\ (\lambda * t)\ \mathbf{then}\ u_0\ \mathbf{else}\ u_1$$
$$\equiv \lambda * (\mathbf{if}^\circ\ (\lambda * t)\ \mathbf{then}\ u_0\ \mathbf{else}\ u_1)$$

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(if°)

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(superpositions)

$$\begin{aligned} t + u &\equiv u + t \\ t + \overrightarrow{0} &\equiv t \\ t + (u + v) &\equiv (t + u) + v \\ \lambda * (t + u) &\equiv \lambda * t + \lambda * u \\ \lambda * t + \kappa * t &\equiv (\lambda + \kappa) * t \\ 0 * t &\equiv \overrightarrow{0} \end{aligned}$$

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$$\begin{aligned} q^{\mathcal{Q}_2}\,\overrightarrow{v} &= (\overrightarrow{v}\,1) * \mathrm{true} + (\overrightarrow{v}\,0) * \mathrm{false} \\ q^{\sigma\otimes\tau}\,\overrightarrow{v} &= \ldots \end{aligned}$$

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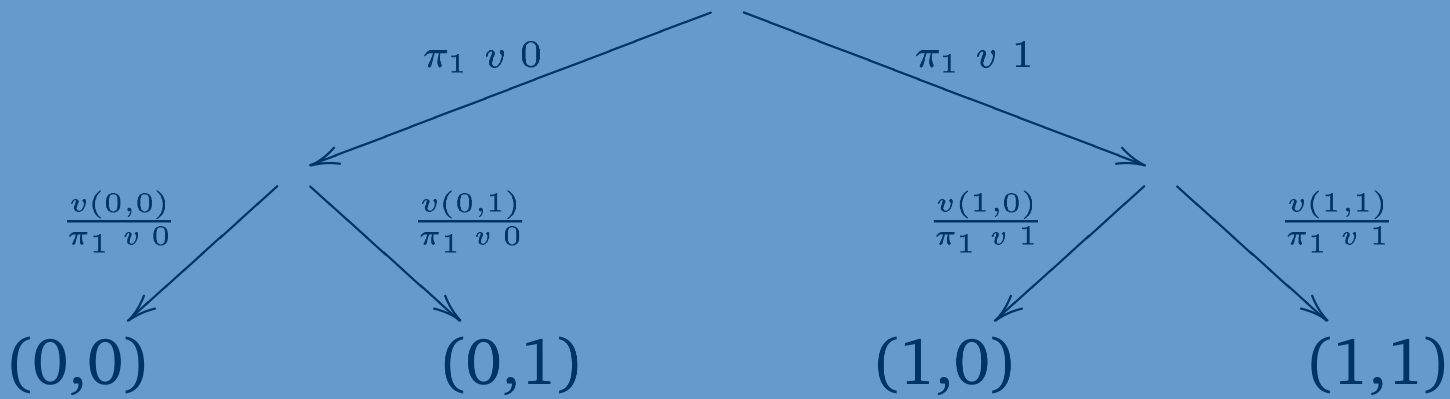

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We need

Lemma
q^σ is linear and isometric and preserves $\otimes$, that is:

1. $q^\sigma\ (\kappa * \overrightarrow{v}) \equiv \kappa * (q^\sigma\ \overrightarrow{v})$
2. $q^\sigma\ (v + w) \equiv (q^\sigma\ v) + (q^\sigma\ w)$
3. $\langle v | w \rangle = \langle q^\sigma\ v | q^\sigma\ w \rangle$
4. $q^{\sigma \otimes \tau}\ v \otimes w \equiv (q^\sigma\ v, q^\tau\ w)$

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Lemma: Inversion

$$\frac{\Gamma \vdash^{\mathrm{Q}} t : \sigma}{\Gamma \vdash^{\mathrm{Q}} \mathrm{nf}^{\sigma}_{\Gamma}(t) \equiv t : \sigma}$$

Proposition: Qompleteness

$$\frac{\Gamma \vdash^{\mathrm{Q}} t, u : \sigma \qquad \llbracket t \rrbracket^{\mathrm{Q}} = \llbracket u \rrbracket^{\mathrm{Q}}}{\Gamma \vdash^{\mathrm{Q}} t \equiv u : \sigma}$$

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- Higher order?
- Indexed types and programs?
- Useful to verify real programs?