Homotopy Type Theory

in 10 minutes

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Functional Programming Lab

A new connection
between ...

Geometry + Logic

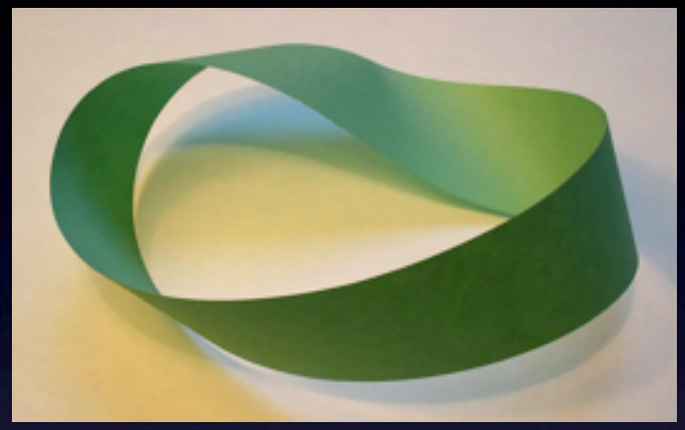
with applications
in Computer
Science!

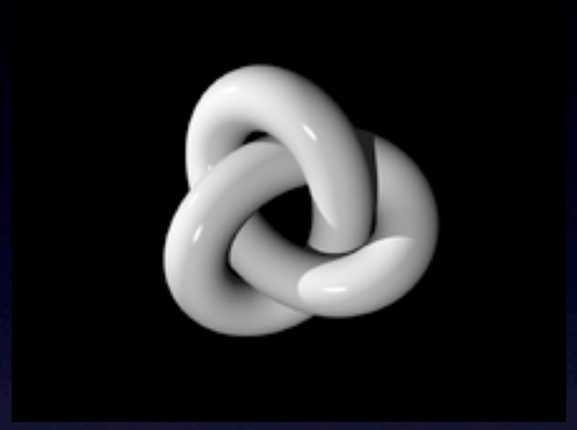
Who ?

Vladimir Voevodsky
(Field medallist)

at the IAS in
Princeton

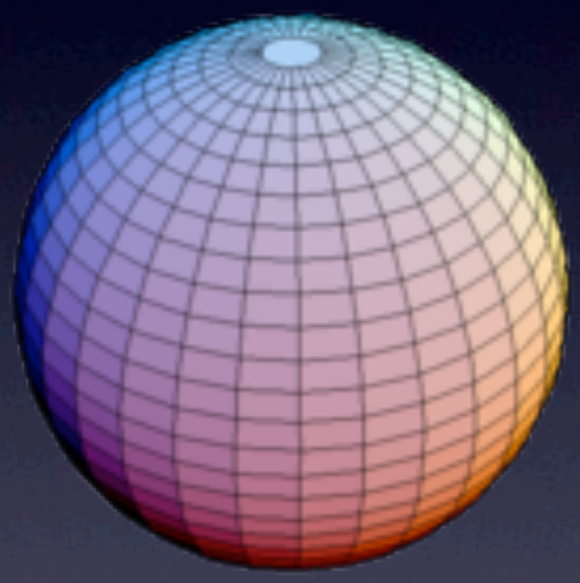
Homotopy Theory ?



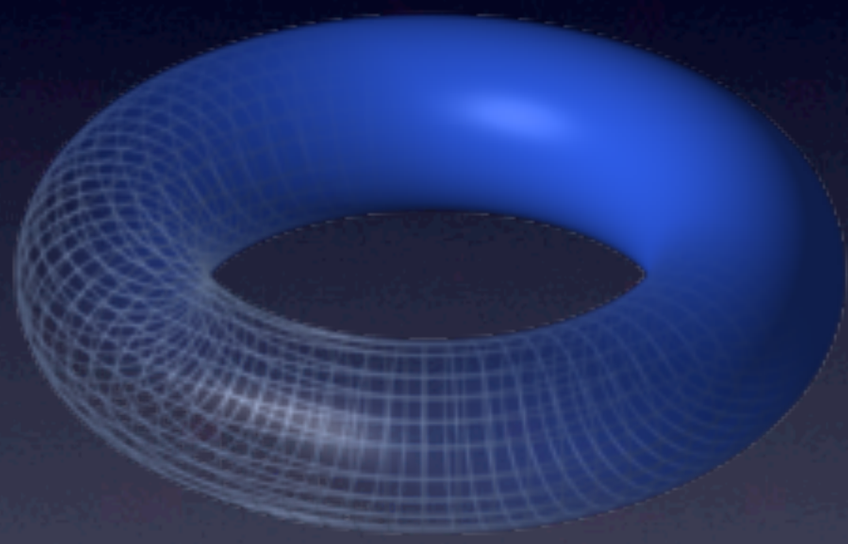


- Related to topology (elastic geometry)
- Classifying geometric objects (spaces) by the groups of paths on them.

What is the difference between



and

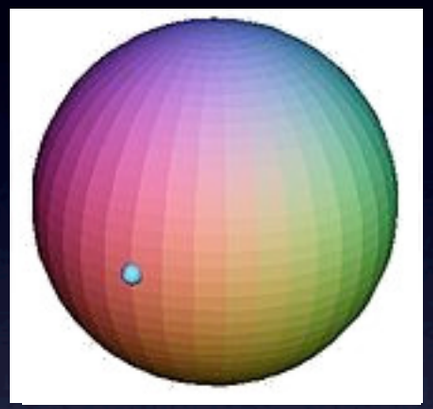


a sphere (S^2)

a torus (T)

?

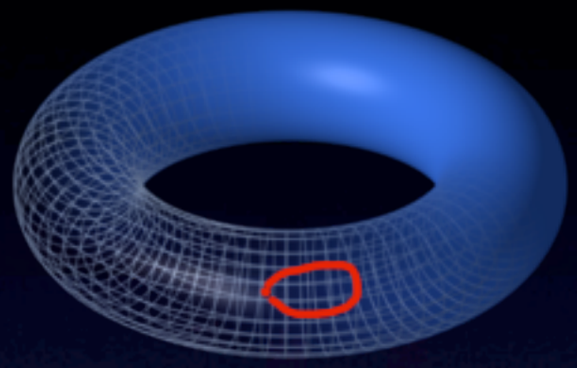
On the sphere

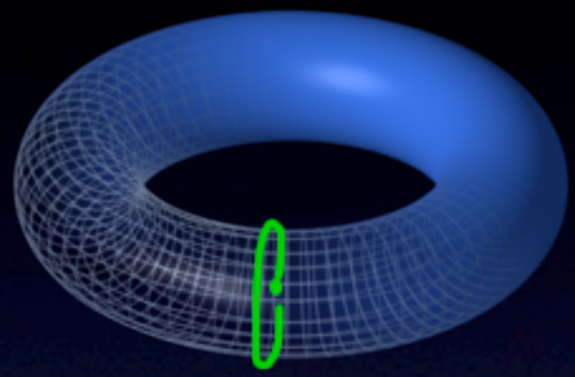


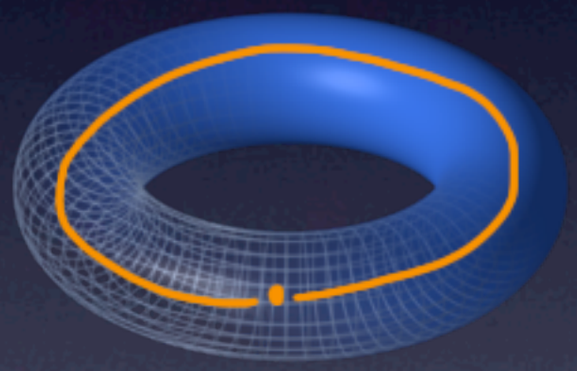
there is only one path from a point to itself

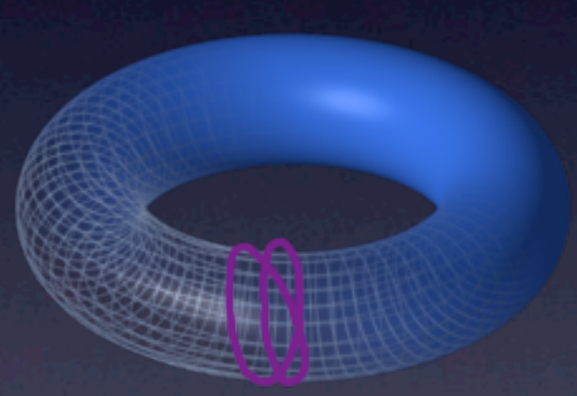
$$\pi_1(S^2) = 0$$

but on the torus





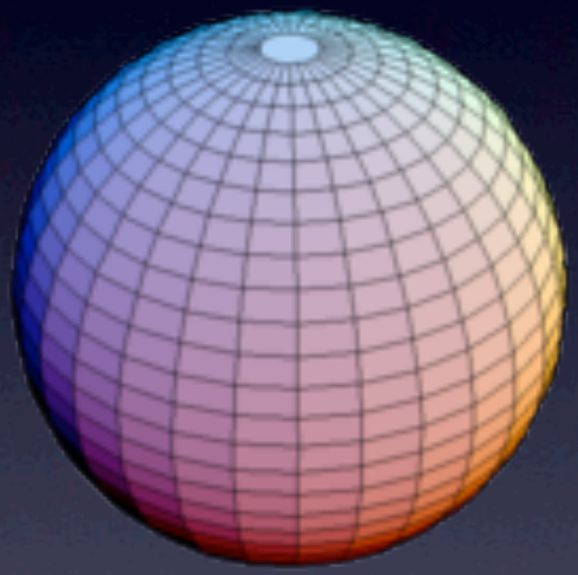




there are lots of different paths

$$\pi_1(T) = \mathbf{Z}^2$$

But: What is the difference between



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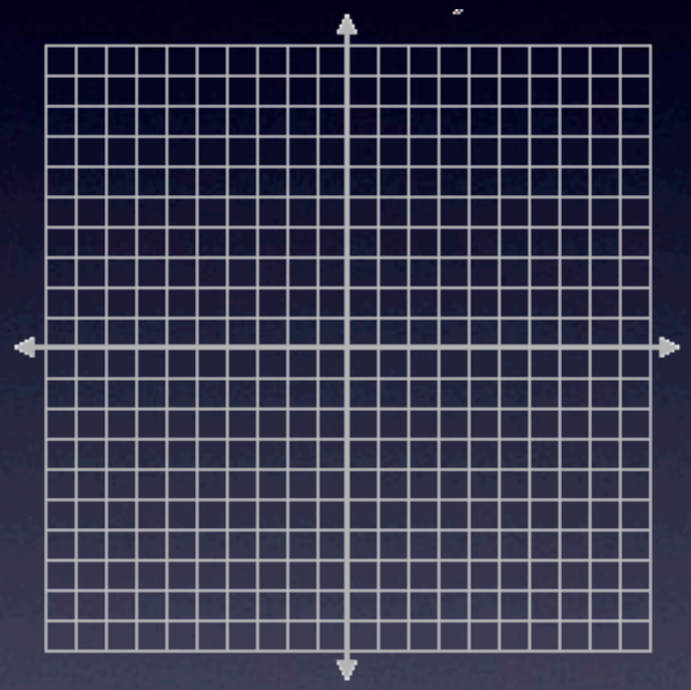


a sphere (S^2)

$$\pi_1(S^2) = 0$$

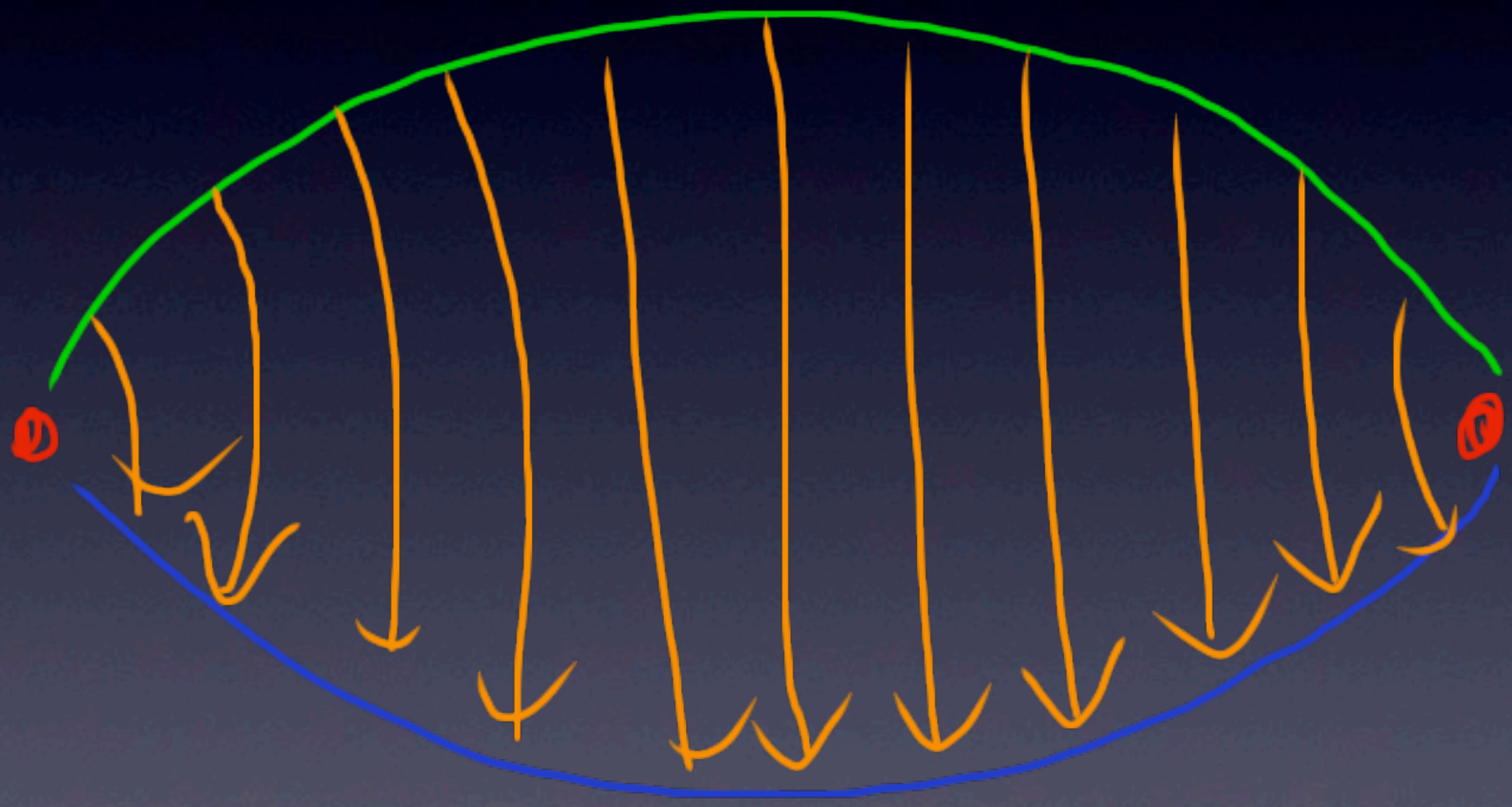
?

the plane (R^2)

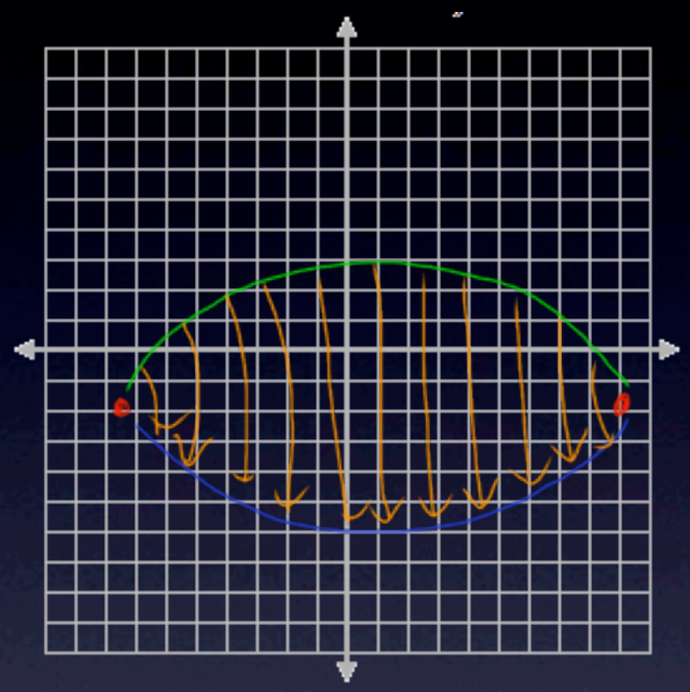
$$\pi_1(\mathbf{R}^2) = 0$$

Higher homotopies

(paths between paths)



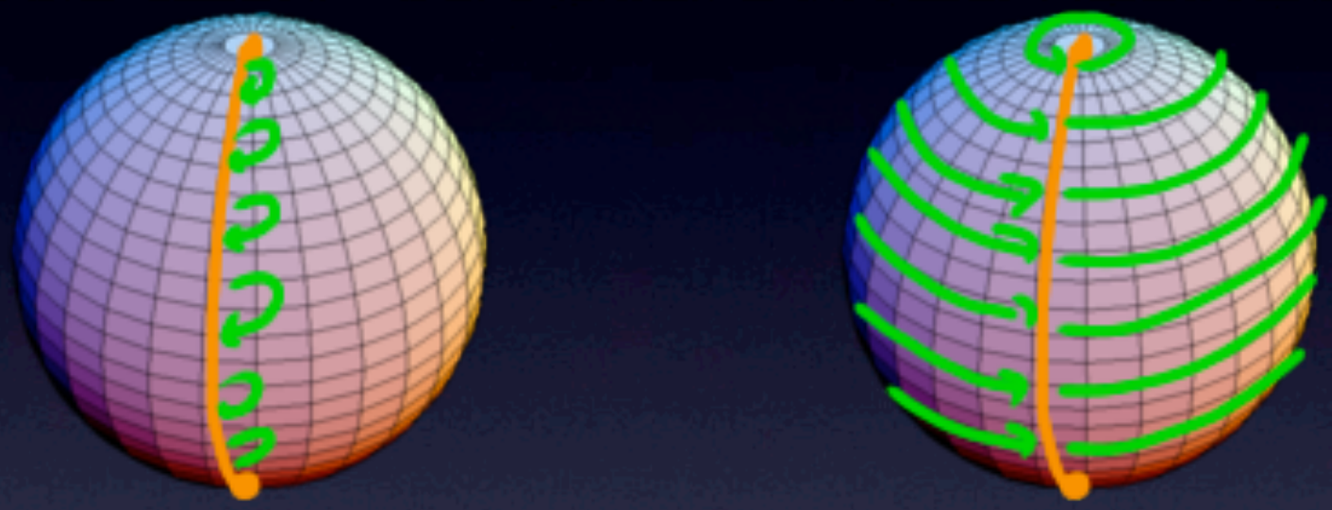
For the plane



there is only 1 path between any 2 paths

$$\pi_2(\mathbf{R}^2) = 0$$

But for the sphere



there are may paths between paths

$$\pi_2(\mathbf{S}^2) = \mathbf{Z}$$

Two geometric objects (spaces) are (homotopically) equivalent iff all their homotopies agree...

Type Theory ?

program	:	type
proof	:	proposition

Curry - Howard - Equivalence

$$A \wedge B \to B \wedge A$$

$$(a, b) \mapsto (b, a)$$

Try Agda !

Equality types

Given $a, b : A$

$$a = b$$

is the type of proofs that a equals b

Higher Equality types

Given $\alpha, \beta : a = b$

$$\alpha = \beta$$

is the type of proofs that two proofs are equal!

How many equality proofs are there ?

There is only one proof that 3 = 3

$$\mathrm{refl} : 3 = 3$$

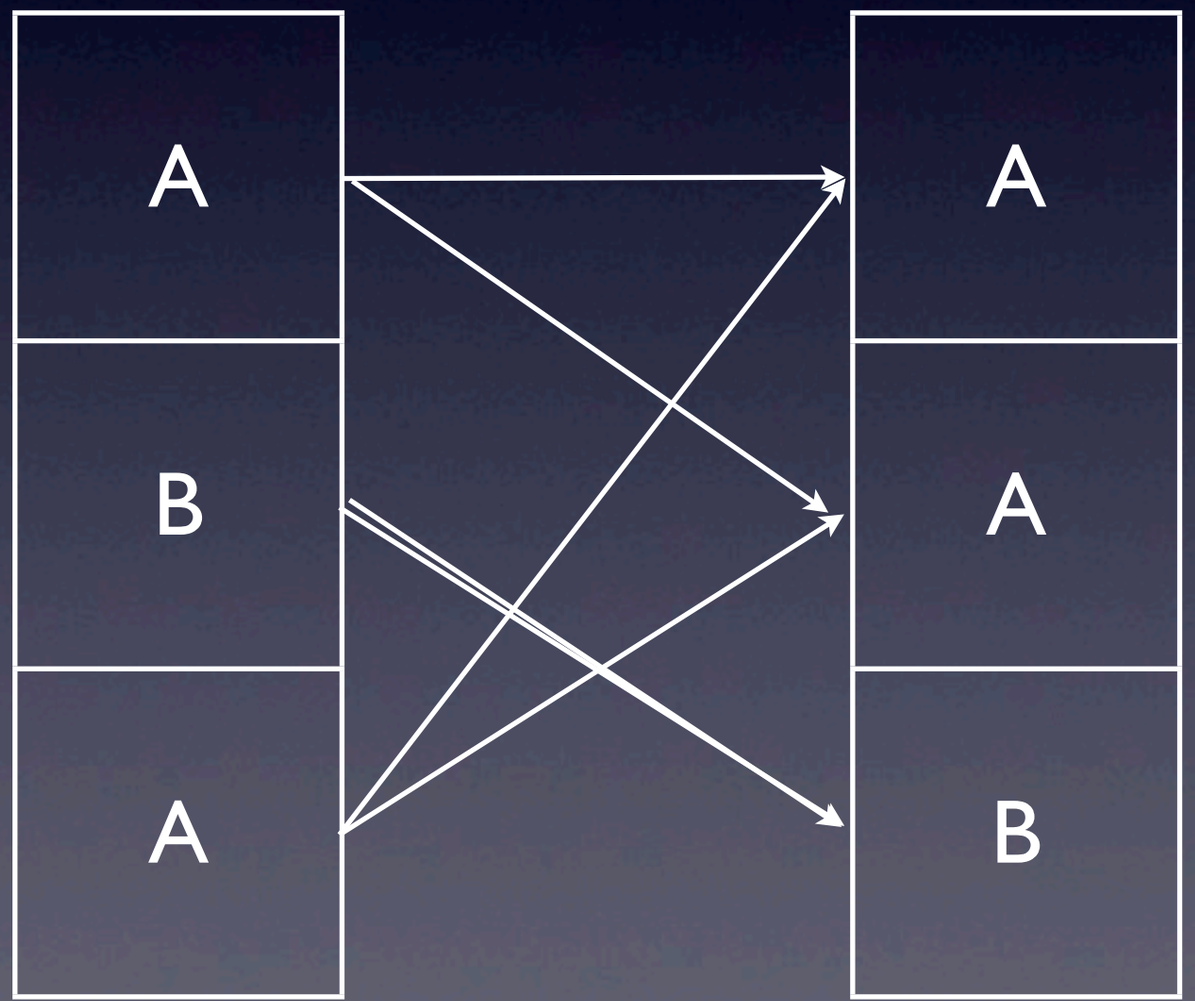
and no proof that

$$0 = 3$$

At most one ?

Equality of data structures ?

$$A \times B \times A = A \times A \times B$$



The connection

- If we want to treat equivalence of datastructures
- e.g. unary numbers = binary numbers
- as equality
- we end up with a theory
- where datatypes behave like spaces
- and equality of datatypes
- is homotopy equaivalence !

Open problems

- Many !
- One of them:
- How to compute in homotopy type theory?

Plan for next Spring