Extensionality now!

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Observational Equality for Dependently Typed Programming

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Shortcomings of Intensional Type Theory (ITT)

Ext is not provable

$$\frac{p \in \Pi a \in A.f\, a = g\, a}{\mathrm{ext}\, p \in f = g}$$

No quotients E.g. $\mathbb{R}$ is not definable.

$$\begin{aligned}
\mathbb{R}_{\mathrm{rep}} &= \{f \in \mathbb{N} \to \mathbb{Q} \mid \\
&\qquad \forall \epsilon > 0.\exists n \in \mathbb{N}.|f\,(n+1) - f\,n| < \epsilon\} \\
\simeq_{\mathbb{R}} &\in \mathbb{R}_{\mathrm{rep}} \to \mathbb{R}_{\mathrm{rep}} \to \mathrm{Prop} \\
f \simeq_{\mathbb{R}} g &= \forall \epsilon > 0.\exists n \in \mathbb{N}.\forall m > n.|f\,m - g\,m| < \epsilon \\
\mathbb{R} &= \mathbb{R}_{\mathrm{rep}} / \simeq_{\mathbb{R}}
\end{aligned}$$

No small core Inductive and coinductive definitions are not reducible to W.
(unlike in Extensional Type Theory).

Asymmetry of ITT

data

- defined by construction.
- **producer contract:** producer promises to only uses legal methods to produce data.
- Examples: Inductive types, e.g. $\mathbb{N}$, finite types, Σ-types, subset types.
- *supported by ITT*

codata

- defined by use
- **consumer contract:** consumer promises only to use legal methods to investigate codata.
- Examples: Coinductive types (e.g. streams), Σ types, Π types, quotient types.
- *not properly supported by ITT*

Per Martin-Löf's classification (MAP 07)

	Excluded middle	Impredicative	Extensional
ZFC set theory	yes	yes	yes
Topos theory	no	yes	yes
Predicative topoi	no	no	yes
ITT	no	no	no

- Is there a foundational issue with extensionality?
- Claim: Extensionality introduces ways of abstraction without increasing the strength of the system.

The goal: Observational Type Theory (OTT)

- ext is provable
- quotients are available
- canonicity holds
- definitional equality ($\equiv$) and type checking are decidable
- definitional proof-irrelevance for propositional types:

$$\frac{P \in \mathrm{Prop} \quad p, q \in P}{p \equiv q}$$

- extends ITT, in particular the definitional equalities for equality elimination hold.

The goal (today)

Implement a universe with extensional equality in ITT (e.g. using Agda or Epigram 1), s.t.

- ext is provable,
- quotient types (like $\mathbb{R}$) are definable,
- canonicity holds for non-propositional types (like $\mathbb{N}$),
- propositional proof irrelevance is provable.

Basic components of the universe

$$\frac{}{\mathrm{U} \in \textbf{Type}} \qquad \frac{A \in \mathrm{U}}{\mathrm{El}\, A \in \textbf{Type}}$$

$$\frac{A, B \in U}{\mathrm{Eq}\, A\, B \in U} \qquad \frac{a \in A \quad b \in B}{\mathrm{eq}\, a\, b \in U}$$

s.t.

$$\frac{p \in \mathrm{El}\,(\mathrm{eq}\,(a \in A)\,(b \in B))}{\mathrm{fog}\, p \in \mathrm{El}\,(\mathrm{Eq}\, A\, B)}$$

Coercion and coherence

$$\frac{p : \mathrm{Eq}\, A_0\, A_1 \qquad a \in \mathrm{El}\, A_0}{\mathrm{coe}\, p\, a \in \mathrm{El}\, A_1}$$

$$\mathrm{coh}\, p\, a \in \mathrm{El}\, (\mathrm{eq}\, a\, (\mathrm{coe}\, p\, a))$$

Example: Π-types

$$\frac{A \in \mathrm{U} \quad B \in (\mathrm{El}\, A) \to \mathrm{U}}{\mathrm{PI}\, A\, B \in \mathrm{U}}$$

$$\frac{f \in \Pi a \in \mathrm{El}\, A.\mathrm{El}\,(B\, a)}{\mathrm{lam}\, f \in \mathrm{El}\,(\mathrm{PI}\, A\, B)}$$

Equality for Π-types

$$\frac{A^{=} \in \mathrm{El}\,(\mathrm{Eq}\,A_1\,A_0) \qquad \dfrac{p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in A_0)\,(a_1 \in A_1))}{B^{=}\,p \in \mathrm{El}\,(\mathrm{Eq}(B_0\,a_0)\,(B_1\,a_1))}}{\mathrm{PI}^{=}\,A^{=}\,B^{=} \in \mathrm{El}(\mathrm{Eq}\,(\mathrm{PI}\,A_0\,B_0)\,(\mathrm{PI}\,A_1\,B_1))}$$

We are cheating!
The official definition of Eq uses only the encoding of types in U, using PI and SIGMA (not given here).

Equality for elements of Π-types

$$\frac{\begin{array}{c} f_i \in \Pi a \in \mathrm{El}\, A_i . \mathrm{El}\,(B_i\, a) \\ A^{=} \in \mathrm{El}\,(\mathrm{Eq}\, A_1\, A_0) \end{array} \quad \dfrac{p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in A_0)\,(a_1 \in A_1))}{f^{=}\, p \in \mathrm{El}\,(\mathrm{eq}\,(f_0\, a_0 \in B\, a_0)\,(f_1\, a_1 \in B\, a_1))}}{\mathrm{lam}^{=}\, A^{=}\, f^{=} \in \mathrm{El}\,(\mathrm{eq}\,(\mathrm{lam}\, f_0)\,(\mathrm{lam}\, f_1))}$$

Coerce for Π-types

$$\frac{\mathrm{PI}^{=}\, A^{=}\, B^{=} : \mathrm{Eq}\,(\mathrm{PI}\, A_0\, B_0)\,(\mathrm{PI}\, A_1\, B_1) \qquad f \in \mathrm{El}\,(\mathrm{PI}\, A_0\, B_0)}{\begin{array}{lcl} \mathrm{coe}\,(\mathrm{PI}^{=}\, A^{=}\, B^{=})\, f & \in & \mathrm{El}\,(\mathrm{PI}\, A_1\, B_1) \\ \mathrm{coe}\,(\mathrm{PI}^{=}\, A^{=}\, B^{=})\, f & = & \lambda a \in A_1 . \mathrm{coe}\,(B^{=}\,(\mathrm{coh}\, A^{=})\, a)\, f\,(\mathrm{coe}\, A^{=}\, a) \\ \\ \mathrm{coh}\,(\mathrm{PI}^{=}\, A^{=}\, B^{=})\, f & \in & \mathrm{El}\,(\mathrm{eq}\, f\,(\mathrm{coe}\,(\mathrm{PI}^{=}\, A^{=}\, B^{=})\, f)) \end{array}}$$

Exercise: Implement coherence.

Other type constructors

- Σ-types
- $0, 1, 2$
  with large elims for 2, e.g. to show $\text{true} \neq \text{false}$.
- W-types
- This collection is sufficent to encode most of everyday Type Theory, including inductive and coinductive families. cf. joint work with Peter Morris et al on Container types.

What about refl?

We cannot prove:

$$\frac{A \in \mathrm{U}}{\mathrm{Refl}\, A \in \mathrm{El}(\mathrm{Eq}\, A\, A)} \qquad \frac{a \in \mathrm{El}\, A}{\mathrm{refl}\, a \in \mathrm{El}\,(\mathrm{eq}\, a\, a)}$$

which actually is a consequence of:

$$\frac{B \in (\mathrm{El}\, A) \to \mathrm{U} \quad p \in \mathrm{El}(\mathrm{eq}\,(a_0 \in A)\,(a_1 \in A))}{\mathrm{Resp}\, B\, p \in \mathrm{El}(\mathrm{Eq}(B\, a_0)\,(B\, a_1))}$$

$$\frac{f \in \Pi a \in \mathrm{El}\, A.\mathrm{El}\,(B\, a) \quad p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in A)\,(a_1 \in A))}{\mathrm{resp}\, f\, p \in \mathrm{El}\,(\mathrm{eq}\,(f\, a_0 \in B\, a_0)\,(f\, a_1 \in B\, a_1))}$$

Hence we are going to assume $\mathrm{Resp}, \mathrm{resp}$!
Are we back at square 1?

Propositional types

We define

$$\mathrm{Prop} \in \mathrm{U} \to \textbf{Type}$$

by

$$\frac{p \in \Pi a, b \in \mathrm{El}\, A.\mathrm{El}\,(\mathrm{eq}\; a\, b)}{\mathrm{prop}\; p \in \mathrm{Prop}\; A}$$

We can show:

$$\frac{A, B \in \mathrm{U}}{\mathrm{Irr} \in \mathrm{Prop}\,(\mathrm{Eq}\; A\, B)} \qquad \frac{a \in \mathrm{El}\, A \quad b \in \mathrm{El}\, B}{\mathrm{irr} \in \mathrm{Prop}\,(\mathrm{eq}\; a\, b)}$$

Observation

- Assumptions in a *consistent* propositional type will only generate non-canonical elements in other propositional types.
- $\mathrm{resp}, \mathrm{Resp}$ are consistent, if ETT is consistent.
- **Hence:** Assuming $\mathrm{resp}, \mathrm{Resp}$ does not destroy canonicity of non-propositional types, like $\mathbb{N}$.

Quotient types ?

- The construction does not work for quotient types, because resp is unsound.
- Instead we define:

$$\frac{A \in \mathrm{U} \quad B \in (\mathrm{El}\,A) \to \mathrm{U} \quad \dfrac{p \in \mathrm{eq}\,(a_0 \in A)\,(a_1 \in A)}{B^{\mathrm{resp}} \in \mathrm{Eq}(B\,a_0)\,(B\,a_1)}}{\mathrm{PI}\,A\,B\,B^{\mathrm{resp}} \in U}$$

$$\frac{\begin{array}{c} f_i \in \Pi a \in \mathrm{El}\,A_i.\mathrm{El}\,(B_i\,a) \\ A^{=} \in \mathrm{El}\,(\mathrm{Eq}\,A_1\,A_0) \end{array} \quad \dfrac{p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in A)\,(a_1 \in A))}{f^{\mathrm{resp}}\,p \in \mathrm{El}\,(\mathrm{eq}\,(f\,a_0)\,(f\,a_1))}}{\mathrm{lam}\,A^{=}\,f\,f^{\mathrm{resp}} \in \mathrm{El}\,(\mathrm{PI}\,A\,B\,B^{\mathrm{resp}})}$$

Quotient types

$$\frac{e \in \mathrm{EqRel}\, R \qquad R \in \mathrm{El}\, A \to \mathrm{El}\, A \to \mathrm{Prop} \qquad \dfrac{\begin{array}{c} p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in A)\,(a_1 \in A)) \\ q \in \mathrm{El}\,(\mathrm{eq}\,(b_0 \in A)\,(b_1 \in A)) \end{array}}{R^{\mathrm{resp}} \in \mathrm{El}\,(\mathrm{Eq}(R_0\, a_0\, b_0)\,(R_1\, a_1\, b_1))}}{\mathrm{Quot}\, A\, R\, R^{\mathrm{resp}}\, e \in \mathrm{U}}$$

$$\frac{a \in \mathrm{El}\, A}{\mathrm{quot}\, a \in \mathrm{El}\,(\mathrm{Quot}\, A\, R\, e)}$$

$$\frac{A^{=} \in \mathrm{El}\,(\mathrm{Eq}\, A_0\, A_1) \qquad \dfrac{\begin{array}{c} p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in A_0)\,(a_1 \in A_1)) \\ q \in \mathrm{El}\,(\mathrm{eq}\,(b_0 \in A_0)\,(b_1 \in A_1)) \end{array}}{R^{=}\, p\, q \in \mathrm{El}\,(\mathrm{Eq}(R_0\, a_0\, b_0)\,(R_1\, a_1\, b_1))}}{\mathrm{Quot}^{=}\, A^{=}\, R^{=} \in \mathrm{El}(\mathrm{Eq}\,(\mathrm{Quot}\, A_0\, R_0\, e_0)\,(\mathrm{Quot}\, A_1\, R_1\, e_1))}$$

Quotient types

$$\frac{a_i \in \mathrm{El}\, A_i \qquad \mathrm{Quot}^{=}\, A^{=}\, R^{=} \in \mathrm{El}(\mathrm{Eq}\,(\mathrm{Quot}\, A_0\, R_0\, e_0)\,(\mathrm{Quot}\, A_1\, R_1\, e_1)) \qquad r \in \mathrm{El}\,(R_1\,(\mathrm{coe}\ A^{=}\, a_0)\, a_1)}{\mathrm{quot}^{=}\, A^{=}\, R^{=}\, r \in \mathrm{eq}\,(\mathrm{quot}\, a_0)\,(\mathrm{quot}\, a_1)}$$

Discrete respect

- We can now prove $\mathrm{refl}, \mathrm{Refl}$ because the elements contain the proofs of resp.
- but now we have to show that functions preserve extensional equality, when introducing them!
- This is necessary for quotients but not for *discrete types*. Hence we should assume:

$$\frac{\begin{array}{c} d \in \mathrm{Discrete}\, A \\ f \in \Pi a \in \mathrm{El}\, A.\mathrm{El}\,(B\, a) \\ p \in \mathrm{El}\,(\mathrm{eq}\,(a_0 \in \mathrm{El}\, A)\,(a_1 \in \mathrm{El}\, A)) \end{array}}{\mathrm{dresp}\, d\, p \in \mathrm{El}\,(\mathrm{eq}\,(f\, a_0)\,(f\, a_1))}$$

- $\mathrm{Discrete} \in \mathrm{U} \to$ **Type** can be defined syntactically (no strictly positive occurence of quotient types).
- Is there a better (intrinsic) characterisation of $\mathrm{Discrete}$?

Discussion

- Our construction does not extend ITT, e.g.

$$\mathrm{coe}\,\mathrm{Refl}\ a \equiv a$$

  doesn't hold, but we can only prove it propositionally.
- We require the consistency of ETT!
- Can we extend this construction to a translation from proof-relevant OTT to ITT?
- Proof-irrelevant OTT is being implemented in Epigram 2.
- We hope to be able to show it's metatheoretic properties directly…
- …using big step normalisation, cf. previous talk.