The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN SEMESTER 2009-2010

MATHEMATICS FOR COMPUTER SCIENTISTS

Time allowed 1 hour and 30 minutes

Candidates must NOT start writing their answers until told to do so

Answer Question 1 and TWO questions of the remaining 4 (Questions 2-5)

Appendices A and B on pg. 8 and pg. 9 give the rules of propositional logic and Boolean algebra

Marks available for sections of questions are shown in brackets in the right-hand margin.

No calculators are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

Question 1: (25)

For the following questions, each correct answer gives 1 mark. Every incorrect or blank answer receives a negative mark of -1.

(a) Given the following three propositional variables, defined as statements in English: (5)

$A =$ *A germ is smaller than a hippopotamus.*
$B =$ *Reality is an illusion.*
$C =$ *The duck clucks.*

Translate the following two English sentences into propositional formulas:

(i) *If a germ is smaller than a hippopotamus, then reality is an illusion.*

(ii) *If reality is an illusion and the duck does not cluck, then a germ is smaller than a hippopotamus.*

(iii) Is the implication connective, $\Rightarrow$, commutative?

(iv) Write the truth table of the propositional formula $A \vee B \Rightarrow \neg A$.

(v) Is the propositional formula $(A \wedge B) \vee A \Rightarrow A$ a tautology?

(b) Compute the values of the following expressions: (5)

(i) $|\lfloor -7.3 \rfloor|$

(ii) $\lfloor |-7.3| \rfloor$

(iii) $\lceil \lfloor 7/2 \rfloor / 2 \rceil$

For each of the following propositions, write if it is true or false:

(iv) For every real number x, $|\lceil x \rceil| = \lceil |x| \rceil$.

(v) For every real number x, $\lfloor x \rfloor = \lceil x \rceil \Rightarrow x = \lfloor x \rfloor$.

(c) For each of the following propositions, write if it is true or false (all variables denote natural numbers): (5)

(i) Every natural number is a divisor of 0.

(ii) Divisibility is a total order relation.

(iii) $(n \mid m) \wedge (k > 0) \Rightarrow (k \cdot n) \mid m$.

(iv) $(i \leq j) \wedge (j + k \leq h + k) \Rightarrow i \leq h$.

(v) The strict order relation, $<$, is reflexive.

(d) Consider the following two sets: (5)

$$A = \{\mathsf{ash}, \mathsf{birch}, \mathsf{maple}, \mathsf{pine}\}$$
$$B = \{\mathsf{birch}, \mathsf{fir}, \mathsf{pine}, \mathsf{oak}\}$$

List the elements of the following sets:

(i) $(A \setminus B) \cup (B \setminus A)$

(ii) $\mathcal{P}(A \cap B)$

For each of the following propositions, write if it is true or false for all sets X and Y:

(iii) $(X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X)$

(iv) $(X \cap Y = X) \Rightarrow Y \subseteq X$

(v) $(Y \setminus X) \cup (X \cap Y) = Y$

(e) Compute the following: (5)

(i) $\sum_{i=0}^{2}(i^2 + 3i)$

(ii) $\sum_{i=3}^{0}(i^2 + i)$

(iii) The number of subsets of $\mathbb{Z}_5$.

(iv) The number of subsets of $\mathbb{Z}_5$ of cardinality 3.

(v) The multiplicative inverse of 3 in $\mathbb{Z}_5$.

Question 2:

This question is about propositional logic and Boolean algebra. (25)

(a) Write the truth table for the following formula and state whether it is a tautology or not: (5)

$$(A \Rightarrow B) \vee (\neg A \wedge \neg C).$$

(b) Complete the following derivation: (10)

$$\begin{array}{l|ll}
1 & A \Rightarrow B & \\
2 & \underline{A \wedge B \Rightarrow C} & \\
\vdots & \vdots & \\
\cdots & \neg(A \wedge \neg C) & \cdots
\end{array}$$

(c) Using Boolean algebra, prove the following propositional equality, justifying every step by one of the rules: (10)

$$(\neg C \wedge A \Rightarrow B) = (A \Rightarrow (\neg B \Rightarrow C)).$$

Question 3:

This question is about recursion and induction. (25)

Consider the recursive function defined as follows:

$$\begin{aligned} \mathsf{oldFun}(0) &= 0 \\ \mathsf{oldFun}(n) &= n^2 - \mathsf{oldFun}(n-1) \qquad \text{if } n > 0 \end{aligned}$$

(a) Compute the following values of oldFun: (5)

$$\begin{aligned} &\mathsf{oldFun}(1) \\ &\mathsf{oldFun}(2) \\ &\mathsf{oldFun}(3) \\ &\mathsf{oldFun}(4) \end{aligned}$$

(b) Prove by induction that the following property holds for every natural number n: (10)

$$P(n): \qquad \mathsf{oldFun}(n) = \frac{n(n+1)}{2}.$$

(c) Complete the following recursive definition (replace the question marks with two natural numbers): (10)

$$\begin{aligned} \mathsf{newFun}(0) &= 0 \\ \mathsf{newFun}(n) &= ? \cdot n^2 - ? \cdot \mathsf{newFun}(n-1) \qquad \text{if } n > 0 \end{aligned}$$

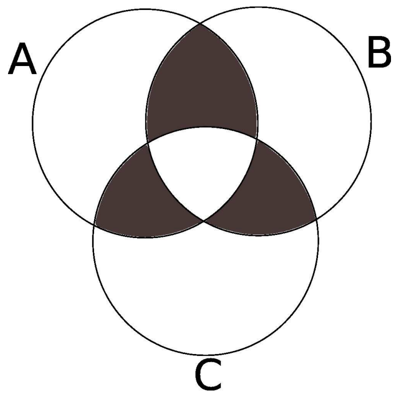
knowing the following values of the function:

$$\begin{aligned} \mathsf{newFun}(1) &= 3 \\ \mathsf{newFun}(2) &= 6 \\ \mathsf{newFun}(3) &= 15 \\ \mathsf{newFun}(4) &= 18 \\ \mathsf{newFun}(5) &= 39 \end{aligned}$$

Question 4:

This question is about sets and functions. (25)

(a) Let A, B and C be three sets. Write a set expression, using the union, intersection and difference operators, that describes the shaded area in the following Venn diagram: (10)



(b) Now take the three sets A, B and C to be defined as follows: (5)

$$A = \{n \in \mathbb{N} \mid 2 \text{ divides } n\}$$
$$B = \{n \in \mathbb{N} \mid 3 \text{ divides } n\}$$
$$C = \{n \in \mathbb{N} \mid \text{there exists a number } m \in \mathbb{N} \text{ such that } n = m^2\}$$

Which of the following numbers belong to the set that you wrote down in part (a)?

$$3, 4, 6, 9, 13, 24, 25, 36, 81$$

(c) Consider the function: (10)

$$f : \mathbb{Z}_5 \to \mathbb{Z}_5$$
$$f(x) = x^2 + 1$$

(i) Is the function a bijection?

(ii) If the answer to (i) is 'yes', write down the inverse of f by giving its values on every element of $\mathbb{Z}_5$.

If the answer to (i) is 'no', give a counterexample (either an element of $\mathbb{Z}_5$ that is not in the image of f, or two elements that have the same image through f).

Question 5:

This question is about combinatorics and modular arithmetic. (25)
Consider the following bijective function:

$$
\begin{aligned}
&f : \mathbb{Z}_5 \rightarrow \mathbb{Z}_5 \\
&f(0) = 2 \\
&f(1) = 4 \\
&f(2) = 3 \\
&f(3) = 0 \\
&f(4) = 1
\end{aligned}
$$

(a) Write down the inverse of f, f^{-1}. (5)

(b) Write down a function g such that, for every $x \in \mathbb{Z}_5$, (5)

$$f(x) \oplus g(x) = x.$$

(c) What is the smallest number n such that $f^n = \mathsf{id}$? (5)

(d) Let X be any subset of the natural numbers with exactly 6 elements. Prove that there must be two different elements x and y of X such that $x - y$ is a multiple of 5. (10)

(Hint: consider the differences of elements of X, $x - y$ modulo 5 and use the pigeonhole principle.)

Appendix A: Rules of propositional logic.

$$\begin{array}{l|ll} m & A & \\ n & B & \\ p & A \wedge B & \wedge\mathrm{I},\ m,\ n \end{array}$$

$$\begin{array}{l|ll} m & A \wedge B & \\ p & A & \wedge\mathrm{E},\ m \end{array} \qquad \begin{array}{l|ll} m & A \wedge B & \\ p & B & \wedge\mathrm{E},\ m \end{array}$$

$$\begin{array}{l|ll} m & A & \\ p & A \vee B & \vee\mathrm{I},\ m \end{array} \qquad \begin{array}{l|ll} m & B & \\ p & A \vee B & \vee\mathrm{I},\ m \end{array}$$

$$\begin{array}{l|ll} m & A \vee B & \\ h & \quad\begin{array}{|l} A \\ \hline \end{array} & \\ i & \quad\begin{array}{|l} C \end{array} & \\ j & \quad\begin{array}{|l} B \\ \hline \end{array} & \\ k & \quad\begin{array}{|l} C \end{array} & \\ p & C & \vee\mathrm{E},\ m,\ h\text{–}i,\ j\text{–}k \end{array}$$

$$\begin{array}{l|ll} m & \quad\begin{array}{|l} A \\ \hline \end{array} & \\ n & \quad\begin{array}{|l} B \end{array} & \\ n+1 & A \Rightarrow B & \Rightarrow\mathrm{I},\ m\text{–}n \end{array}$$

$$\begin{array}{l|ll} m & A \Rightarrow B & \\ n & A & \\ p & B & \Rightarrow\mathrm{E},\ m,\ n \end{array}$$

$$\begin{array}{l|ll} m & \quad\begin{array}{|l} A \\ \hline \end{array} & \\ n & \quad\begin{array}{|l} \bot \end{array} & \\ n+1 & \neg A & \neg\mathrm{I},\ m\text{–}n \end{array}$$

$$\begin{array}{l|ll} m & \neg A & \\ n & A & \\ p & \bot & \neg\mathrm{E},\ m,\ n \end{array}$$

$$\begin{array}{l|ll} m & \bot & \\ p & C & \bot\mathrm{E},\ m \end{array}$$

$$\begin{array}{l|ll} m & A & \\ p & A & \mathrm{R},\ m \end{array}$$

m $\mid$ $\neg\neg A$ p $\mid$ A $\neg\neg$E, m	p $\mid$ $A \vee \neg A$ EM

Appendix B: Boolean algebra.

$A \wedge B = B \wedge A$	Commutativity of conjunction
$A \vee B = B \vee A$	Commutativity of disjunction
$A \wedge (B \wedge C) = (A \wedge B) \wedge C$	Associativity of conjunction
$A \vee (B \vee C) = (A \vee B) \vee C$	Associativity of disjunction
$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$	Distributivity of conj. over disj.
$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$	Distributivity of disj. over conj.
$\neg(A \wedge B) = \neg A \vee \neg B$	First De Morgan law
$\neg(A \vee B) = \neg A \wedge \neg B$	Second De Morgan law
$A \wedge \mathsf{true} = A$	Unit of conjunction
$A \vee \mathsf{false} = A$	Unit of disjunction
$A \wedge \mathsf{false} = \mathsf{false}$	Zero of conjunction
$A \vee \mathsf{true} = \mathsf{true}$	Zero of disjunction
$A \wedge A = A$	Idempotence of conjunction
$A \vee A = A$	Idempotence of disjunction
$A \wedge (A \vee B) = A$	First absorption law
$A \vee (A \wedge B) = A$	Second absorption law
$A \wedge \neg A = \mathsf{false}$	Contradiction
$\neg\neg A = A$	Double negation
$A \vee \neg A = \mathsf{true}$	Excluded middle
$A \Rightarrow B = \neg A \vee B$	Definition of implication