G52DOA - Derivation of Algorithms
Propositional Logic

Venanzio Capretta

Propositions

A *proposition* is any statement that may be true or false:

- "Nottingham is a town in England"
- "The moon is made of cheese"
- "$1 + 1 = 2$"
- "$7 < 3$"
- "There is intelligent life in the Andromeda galaxy"

We use capital letters $A, B, C, \ldots$ to denote propositions.

Connectives are operations to combine simple propositions into complex ones:

- Conjunction: "and" ($\wedge$);
- Disjunction: "or" ($\vee$);
- Implication: "if ... then ..." ($\rightarrow$);
- Negation: "not" ($\neg$).

We also have two basic propositions, one true, the other false:

- "True" ($\top$): "$0 = 0$";
- "False" ($\bot$): "$0 = 1$".

When you construct a complex proposition with a connective, you are supposed to put parentheses around it.

Examples, let A, B, C be *atomic* propositions.

$$(A \wedge B)$$
$$((A \wedge B) \wedge C)$$
$$((A \wedge B) \rightarrow (A \vee B))$$
$$((A \rightarrow B) \rightarrow ((\neg B) \rightarrow (\neg A)))$$
$$((A \rightarrow (B \rightarrow (C \rightarrow D))) \rightarrow (((A \wedge B) \wedge C) \rightarrow D))$$

Too many parentheses! How can we simplify the notation?

Associativity rules:

- Conjunction associates to the left:
  $A \wedge B \wedge C$ means $((A \wedge B) \wedge C)$;
- Disjunction associates to the left:
  $A \vee B \vee C$ means $((A \vee B) \vee C)$;
- Implication associates to the right:
  $A \to B \to C$ means $(A \to (B \to C))$.

Precedence rules. The order of precedence of the connectives is:

$$\neg,\ \wedge\ ,\ \vee\ ,\ \to\ .$$

This means that:

- $\neg A \wedge B$ means $((\neg A) \wedge B)$ (negation has precedence over conjunction);
- $A \vee B \wedge C$ means $(A \vee (B \wedge C))$;
- $A \to B \vee C$ means $(A \to (B \vee C))$.

The examples given earlier become:

$$A \wedge B$$
$$A \wedge B \wedge C$$
$$A \wedge B \to A \vee B$$
$$(A \to B) \to \neg B \to \neg A$$
$$(A \to B \to C \to D) \to A \wedge B \wedge C \to D$$

CAUTION: sometimes the parentheses are needed. In the forth and fifth examples above, I cannot delete them without changing the meaning of the proposition.

Propositional Logic: Rules of Natural Deduction

To prove that some propositions are true we use a system of *natural deduction.*

Derivations are written as lists of propositions.

Every proposition is the consequence of preceding ones.

Except for *assumptions* which are assumed without proof.

1	A	assumption
2	B	justification
$\vdots$		
n	Z	justification (conclusion)

Sometimes we use boxes to make some *local assumptions.*

Propositions inside a box are *visible* only from within the box.

$$
\begin{array}{|l}
A \\
\boxed{\begin{array}{c} B \\ C \end{array}} \\
D \\
F
\end{array}
$$

To prove F, I can use A and D, but **not** B and C.

D can be a conclusion of *the whole derivation* inside the box containing B and C.

Conjunction

Introduction

$$
\begin{array}{r|ll}
 & \vdots & \\
n & A & \\
 & \vdots & \\
m & B & \\
 & \vdots & \\
k & A \wedge B & \wedge_i \; n, m
\end{array}
$$

Elimination

$$
\begin{array}{r|ll}
 & \vdots & \\
n & A \wedge B & \\
 & \vdots & \\
k & A & \wedge_{e1} \; n
\end{array}
\qquad
\begin{array}{r|ll}
 & \vdots & \\
n & A \wedge B & \\
 & \vdots & \\
k & B & \wedge_{e2} \; n
\end{array}
$$

Disjunction

Introduction

$$
\begin{array}{r|ll}
 & \vdots & \\
n & A & \\
 & \vdots & \\
k & A \vee B & \vee_{i1} \; n
\end{array}
\qquad
\begin{array}{r|ll}
 & \vdots & \\
n & B & \\
 & \vdots & \\
k & A \vee B & \vee_{i2} \; n
\end{array}
$$

Elimination

$$
\begin{array}{r|ll}
 & \vdots & \\
h & A \vee B & \\
 & \vdots & \\
n & \quad A & \text{assumption} \\
 & \quad \vdots & \\
m & \quad C & \\
p & \quad B & \text{assumption} \\
 & \quad \vdots & \\
q & \quad C & \\
 & \vdots & \\
k & C & \vee_e \; h, n-m, p-q
\end{array}
$$

Implication

Introduction

	$\vdots$	
n	$\quad A$	assumption
	$\quad \vdots$	
m	$\quad B$	
	$\vdots$	
k	$A \to B$	$\to_i \; n - m$

Elimination (Modus Ponens)

	$\vdots$	
n	$A \to B$	
	$\vdots$	
m	A	
	$\vdots$	
k	B	$\to_e \; n, m$

Negation ($\neg A \equiv A \to \bot$)

Introduction

	$\vdots$	
n	$\quad A$	assumption
	$\quad \vdots$	
m	$\quad \bot$	
	$\vdots$	
k	$\neg A$	$\neg_i \; n - m$

Elimination

	$\vdots$	
n	$\neg A$	
	$\vdots$	
m	A	
	$\vdots$	
k	$\bot$	$\neg_e \; n, m$

Falsity

No Introduction

Elimination

	$\vdots$	
n	$\bot$	
	$\vdots$	
k	A	$\bot_e \; n$

Truth

Introduction

	$\vdots$	
k	$\top$	$\top_i \; k$

No Elimination

Classical Logic: Double negation elimination

	$\vdots$	
n	$\neg\neg A$	
	$\vdots$	
k	A	$\neg\neg_e \; n$

Truth Tables

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

A	B	$A \to B$
1	1	1
1	0	0
0	1	1
0	0	1

A	$\neg A$
1	0
0	1

A proposition is a *tautology* (i.e. it is provable) if and only if its truth table gives always 1 for every row.

Example: $(\neg A \vee \neg B) \to \neg(A \wedge B)$ is a tautology. Here is its truth table:

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$A \wedge B$	$\neg(A \wedge B)$	$(\neg A \vee \neg B) \to \neg(A \wedge B)$
1	1	0	0	0	1	0	1
1	0	0	1	1	0	1	1
0	1	1	0	1	0	1	1
0	0	1	1	1	0	1	1