

G52DOA - Derivation of Algorithms

Propositional Logic

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Propositions

A *proposition* is any statement that may be true or false:

- “Nottingham is a town in England”
- “The moon is made of cheese”
- “ $1 + 1 = 2$ ”
- “ $7 < 3$ ”
- “There is intelligent life in the Andromeda galaxy”

We use capital letters A, B, C, \dots to denote propositions.

Connectives are operations to combine simple propositions into complex ones:

- Conjunction: “and” (\wedge);
- Disjunction: “or” (\vee);
- Implication: “if ... then ...” (\rightarrow);
- Negation: “not” (\neg).

We also have two basic propositions, one true, the other false:

- “True” (\top): “ $0 = 0$ ”;
- “False” (\perp): “ $0 = 1$ ”.

When you construct a complex proposition with a connective, you are supposed to put parentheses around it.

Examples, let A, B, C be *atomic* propositions.

$(A \wedge B)$
 $((A \wedge B) \wedge C)$
 $((A \wedge B) \rightarrow (A \vee B))$
 $((A \rightarrow B) \rightarrow ((\neg B) \rightarrow (\neg A)))$
 $((A \rightarrow (B \rightarrow (C \rightarrow D))) \rightarrow (((A \wedge B) \wedge C) \rightarrow D))$

Too many parentheses! How can we simplify the notation?

Associativity rules:

- Conjunction associates to the left:
 $A \wedge B \wedge C$ means $((A \wedge B) \wedge C)$;
- Disjunction associates to the left:
 $A \vee B \vee C$ means $((A \vee B) \vee C)$;
- Implication associates to the right:
 $A \rightarrow B \rightarrow C$ means $(A \rightarrow (B \rightarrow C))$.

Precedence rules. The order of precedence of the connectives is:

$$\neg, \wedge, \vee, \rightarrow.$$

This means that:

- $\neg A \wedge B$ means $((\neg A) \wedge B)$ (negation has precedence over conjunction);
- $A \vee B \wedge C$ means $(A \vee (B \wedge C))$;
- $A \rightarrow B \vee C$ means $(A \rightarrow (B \vee C))$.

The examples given earlier become:

$$\begin{aligned} &A \wedge B \\ &A \wedge B \wedge C \\ &A \wedge B \rightarrow A \vee B \\ &(A \rightarrow B) \rightarrow \neg B \rightarrow \neg A \\ &(A \rightarrow B \rightarrow C \rightarrow D) \rightarrow A \wedge B \wedge C \rightarrow D \end{aligned}$$

CAUTION: sometimes the parentheses are needed. In the forth and fifth examples above, I cannot delete them without changing the meaning of the proposition.

Propositional Logic: Rules of Natural Deduction

To prove that some propositions are true we use a system of *natural deduction*.

Derivations are written as lists of propositions.

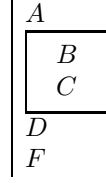
Every proposition is the consequence of preceding ones.

Except for *assumptions* which are assumed without proof.

$$\begin{array}{l|l} 1 & A \text{ assumption} \\ 2 & B \text{ justification} \\ \vdots & \\ n & Z \text{ justification (conclusion)} \end{array}$$

Sometimes we use boxes to make some *local assumptions*.

Propositions inside a box are *visible* only from within the box.

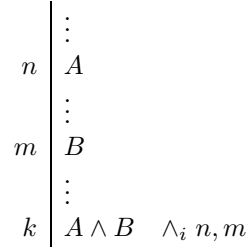


To prove F , I can use A and D , but **not** B and C .

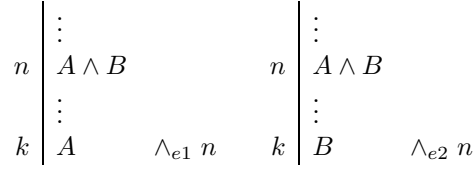
D can be a conclusion of *the whole derivation* inside the box containing B and C .

Conjunction

Introduction

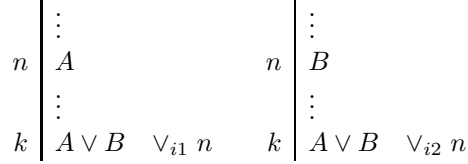


Elimination

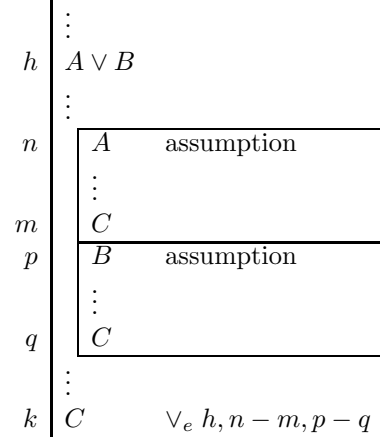


Disjunction

Introduction



Elimination



Implication

Introduction

	\vdots		
n	A	assumption	
	\vdots		
m	B		
	\vdots		
k	$A \rightarrow B$	\rightarrow_i	$n - m$

Elimination (Modus Ponens)

	\vdots		
n	$A \rightarrow B$		
	\vdots		
m	A		
	\vdots		
k	B	\rightarrow_e	n, m

Negation ($\neg A \equiv A \rightarrow \perp$)

Introduction

	\vdots		
n	A	assumption	
	\vdots		
m	\perp		
	\vdots		
k	$\neg A$	\neg_i	$n - m$

Elimination

	\vdots		
n	$\neg A$		
	\vdots		
m	A		
	\vdots		
k	\perp	\neg_e	n, m

Falsity

No Introduction

Elimination

	\vdots		
n	\perp		
	\vdots		
k	A	\perp_e	n

Truth

Introduction

	\vdots		
k	\top	\top_i	k

No Elimination

Classical Logic: Double negation elimination

	\vdots		
n	$\neg\neg A$		
	\vdots		
k	A	$\neg\neg_e$	n

Truth Tables

A	B	$A \wedge B$	A	B	$A \vee B$	A	B	$A \rightarrow B$	A	$\neg A$
1	1	1	1	1	1	1	1	1	1	0
1	0	0	1	0	1	1	0	0	0	1
0	1	0	0	1	1	0	1	1		
0	0	0	0	0	0	0	0	1		

A proposition is a *tautology* (i.e. it is provable) if and only if its truth table gives always 1 for every row.

Example: $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$ is a tautology. Here is its truth table:

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$A \wedge B$	$\neg(A \wedge B)$	$(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$
1	1	0	0	0	1	0	1
1	0	0	1	1	0	1	1
0	1	1	0	1	0	1	1
0	0	1	1	1	0	1	1