The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 2 MODULE, SPRING SEMESTER 2008-2009

DERIVATION OF ALGORITHMS

Time allowed TWO hours

Candidates must NOT start writing their answers until told to do so

Answer FOUR out of six questions

Marks available for sections of questions are shown in brackets in the right-hand margin.

No calculators are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

Question 1:

Let E be a binary relation symbol.
Is the following proposition a tautology?

$$\forall x, \forall y, (\mathsf{E}(x,y) \to \exists z, (z \neq x \land z \neq y \land \mathsf{E}(x,z) \land \mathsf{E}(y,z))) \\ \to \forall x, \forall y, (\mathsf{E}(x,y) \to \mathsf{E}(y,x)).$$

If you think that it is a tautology, give a full derivation of it in Fitch-style natural deduction.
If you think that it is not a tautology, give a graph countermodel of it, where the atomic proposition $\mathsf{E}(x,y)$ is interpreted as *"there is an edge from node x to node y"*; Write the truth table for E in your countermodel.

(25)

Question 2:

Let E be a binary relation symbol and G a unary predicate symbol.
Is the following proposition a tautology?

$$\forall x, ((\mathsf{G}(x) \to \exists y, \mathsf{E}(x,y)) \to \mathsf{G}(x)) \to \forall z, \mathsf{G}(z).$$

If you think that it is a tautology, give a full derivation of it in Fitch-style natural deduction.
If you think that it is not a tautology, give a graph countermodel of it, where the atomic proposition $\mathsf{E}(x,y)$ is interpreted as *"there is an edge from node x to node y"* and the atomic proposition $\mathsf{G}(x)$ is interpreted as *"the node x is green"*; Write the truth tables for E and G in your countermodel.

(25)

Question 3:

Prove, by induction on the natural number n, that the following proposition is true:

$$\forall n \geq 1,\ 3 \cdot \sum_{i=0}^{n-1} i(i+1) + n = n^3.$$

(25)

Question 4:

a) Compute the weakest precondition P of the following program with respect to the given postcondition:

$$\{P\} \left| \begin{array}{l} \mathsf{z} := \mathsf{x} + \mathsf{y}; \\ \mathsf{if}\ \mathsf{x} < \mathsf{y}\ \mathsf{then} \\ \quad \mathsf{x} := \mathsf{y} - \mathsf{x} \\ \mathsf{else} \\ \quad \mathsf{x} := \mathsf{x} - \mathsf{y} \\ \ ; \\ \mathsf{y} := (\mathsf{x} + \mathsf{z})/2 \end{array} \right| \{\mathsf{y} = \max(x_0, y_0)\}$$

(20)

b) Prove that the weakest precondition P that you found is implied by the precondition $\{\mathsf{x} = x_0 \wedge \mathsf{y} = y_0\}$. (5)

Question 5:

a) Give a proof of correctness of the following program with respect to the given specification, by completing the proof tableau:

	$\{\mathsf{x} \geq 0\}$
$\mathsf{z} := 0;$	
$\mathsf{y} := 0;$	
$\mathsf{while}\ \mathsf{z} < \mathsf{x}\ \mathsf{do}$ (	
$\quad \mathsf{y} := \mathsf{y} + \mathsf{z};$	
$\quad \mathsf{z} := \mathsf{z} + 1;$	
$\quad \mathsf{y} := \mathsf{y} + \mathsf{z}$	
$\quad$)	$\{\mathsf{y} = \mathsf{x}^2\}$

(15)

b) Write the non-trivial logical implications that still need to be proved to have a complete proof of correctness according to Hoare logic. (5)

c) Write a *variant* expression for the while loop. Explain how to use it to prove that the loop terminates. (5)

Question 6:

a) You are going to construct a program that satisfies a given specification. The input variables are a natural number n and an array of integers $\vec{\mathsf{a}}$ of length n. The output variable is an array of integers $\vec{\mathsf{b}}$ also of length n. Construct a program P that satisfies the following Hoare triple:

$$\{\top\}\ P\ \{\forall i, 0 \leq i < n \to \mathsf{b}_i = \min\{\mathsf{a}_0, \ldots \mathsf{a}_i\}\}$$

You are not allowed to modify the input variables n and $\vec{\mathsf{a}}$ inside the program P. You can assume that the binary minimum operation, $\min(x, y)$ is given. (15)

b) What are the *invariant* and the *variant* for the loop of the program that you wrote in part (a)? (10)