Solution to the Model Exam for G52DOA, 2009

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Question 1

(a) If we translate the premises and the conclusion in terms of nodes and edges of a graph, we obtain:

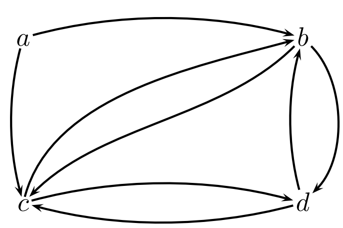
$\forall x, \forall y, (\mathsf{E}(x, y) \rightarrow \exists z, (z \neq x \wedge z \neq y \wedge \mathsf{E}(x, z) \wedge \mathsf{E}(y, z)))$

For every edge with source x and target y, there is a third node z, different from x and y, with edges pointing from x and y to it.

$\forall x, \forall y, (\mathsf{E}(x, y) \rightarrow \mathsf{E}(y, x))$

For every edge there is another edge between the same nodes going in the opposite direction.

(b) The proposition is not a tautology. The following graph satisfies the assumption and falsifies the conclusion:



Here is the truth table for E in the counterexample:

x	y	$\mathsf{E}(x,y)$	x	y	$\mathsf{E}(x,y)$
a	a	$\bot$	c	a	$\bot$
	b	$\top$		b	$\top$
	c	$\top$		c	$\bot$
	d	$\bot$		d	$\top$
b	a	$\bot$	d	a	$\bot$
	b	$\bot$		b	$\top$
	c	$\top$		c	$\top$
	d	$\top$		d	$\bot$

Question 2

The proposition is a tautology. Here is its derivation:

1	$\forall x, ((\mathsf{G}(x) \to \exists y, \mathsf{E}(x,y)) \to \mathsf{G}(x))$	assumption
	a	eigenvariable
2	$(\mathsf{G}(x) \to \exists y, \mathsf{E}(x,y)) \to \mathsf{G}(x)$	$\forall_e 1$
3	$\neg \mathsf{G}(a)$	assumption
4	$\mathsf{G}(a)$	assumption
5	$\bot$	$\neg_e 3, 4$
6	$\exists y, \mathsf{E}(x,y)$	$\bot_e 5$
7	$\mathsf{G}(a) \to \exists y, \mathsf{E}(x,y)$	$\to_i 4-6$
8	$\mathsf{G}(a)$	$\to_e 2, 7$
9	$\bot$	$\neg_e 3, 8$
10	$\neg\neg \mathsf{G}(a)$	$\neg_i 3-9$
11	$\mathsf{G}(a)$	$\neg\neg_e 10$
12	$\forall z, \mathsf{G}(z)$	$\forall_i 2-11$
13	$\forall x, ((\mathsf{G}(x) \to \exists y, \mathsf{E}(x,y)) \to \mathsf{G}(x)) \to \forall z, \mathsf{G}(z)$	$\to_i 1-12$

Question 3

Base case: $n = 1$

$$3 \cdot \sum_{i=0}^{0} i(i+1) + 0 = 3 \cdot 0(0+1) = 0 = 0^3.$$

Induction step: Induction Hypothesis:

$$3 \cdot \sum_{i=0}^{n-1} i(i+1) + n = n^3.$$

To prove:

$$3 \cdot \sum_{i=0}^{n} i(i+1) + n + 1 = (n+1)^3.$$

In fact:

$$\begin{aligned} 3 \cdot \textstyle\sum_{i=0}^{n} i(i+1) + n + 1 &= 3 \cdot \textstyle\sum_{i=0}^{n-1} i(i+1) + 3 \cdot n(n+1) + n + 1 \\ &\overset{\text{I.H}}{=} n^3 + 3 \cdot n(n+1) + 1 = n^3 + 3n^2 + 3n + 1 \\ &= (n+1)^3. \end{aligned}$$

Question 4

(a)

$$
\begin{array}{r|l|l}
\{P\} & \mathsf{z := x + y;} & \\
\{(\mathsf{x < y} \to R_1) \wedge (\neg \mathsf{x < y} \to R_2)\} & \textsf{if } \mathsf{x < y} \textsf{ then} & \\
\{R_1\} & \quad \mathsf{x := y - x} & \{(\mathsf{x + z})/2 = \max(x_0, y_0)\} \\
 & \textsf{else} & \\
\{R_2\} & \quad \mathsf{x := x - y} & \{(\mathsf{x + z})/2 = \max(x_0, y_0)\} \\
 & ; & \\
\{(\mathsf{x + z})/2 = \max(x_0, y_0)\} & \mathsf{y := (x + z)/2} & \{\mathsf{y} = \max(x_0, y_0)\}
\end{array}
$$

where

$$
\begin{array}{rl}
R_1 & \equiv ((\mathsf{x + z})/2 = \max(x_0, y_0))[\mathsf{y - x/x}] \\
 & \equiv (\mathsf{y - x + z})/2 = \max(x_0, y_0) \\
R_2 & \equiv ((\mathsf{x + z})/2 = \max(x_0, y_0))[\mathsf{x - y/x}] \\
 & \equiv (\mathsf{x - y + z})/2 = \max(x_0, y_0) \\
P & \equiv ((\mathsf{x < y} \to R_1) \wedge (\neg \mathsf{x < y} \to R_2))[\mathsf{x + y/z}] \\
 & \equiv (\mathsf{x < y} \to (\mathsf{y - x + x + y})/2 = \max(x_0, y_0)) \wedge (\neg \mathsf{x < y} \to (\mathsf{x - y + x + y})/2 = \max(x_0, y_0)) \\
 & \equiv (\mathsf{x < y} \to \mathsf{y} = \max(x_0, y_0)) \wedge (\neg \mathsf{x < y} \to \mathsf{x} = \max(x_0, y_0))
\end{array}
$$

(b) Assume that $\mathsf{x} = x_0 \wedge \mathsf{y} = y_0$. Then P becomes, by substitution of equals for equals (equality elimination):

$$(x_0 < y_0 \to y_0 = \max(x_0, y_0)) \wedge (\neg x_0 < y_0 \to x_0 = \max(x_0, y_0))$$

which is true by definition of max.

Question 5

(a)

$$
\begin{array}{r|l|l}
 & & \{\mathsf{x} \geq 0\} \\
\{0 = 0^2 \wedge 0 \leq \mathsf{x}\} & \mathsf{z := 0;} & \\
\{0 = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x}\} & \mathsf{y := 0;} & \{\mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x}\} \\
\{\textsf{invariant} : \mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x}\} & \textsf{while } \mathsf{z < x} \textsf{ do (} & \{\mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x} \wedge \mathsf{z} < \mathsf{x}\} \\
\{\mathsf{y} + 2\mathsf{z} + 1 = (\mathsf{z} + 1)^2 \wedge \mathsf{z} + 1 \leq \mathsf{x}\} & \quad \mathsf{y := y + z;} & \\
\{\mathsf{y} + \mathsf{z} + 1 = (\mathsf{z} + 1)^2 \wedge \mathsf{z} + 1 \leq \mathsf{x}\} & \quad \mathsf{z := z + 1;} & \\
\{\mathsf{y} + \mathsf{z} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x}\} & \quad \mathsf{y := y + z} & \{\mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x}\} \\
 &) & \{\mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x} \wedge \neg \mathsf{z} < \mathsf{x}\} \\
 & & \{\mathsf{y} = \mathsf{x}^2\}
\end{array}
$$

(b)

$$\mathsf{x} \geq 0 \to 0 = 0^2 \wedge 0 \leq \mathsf{x}$$

$$\mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x} \wedge \mathsf{z} < \mathsf{x} \to \mathsf{y} + 2\mathsf{z} + 1 = (\mathsf{z} + 1)^2 \wedge \mathsf{z} + 1 \leq \mathsf{x}$$

$$\mathsf{y} = \mathsf{z}^2 \wedge \mathsf{z} \leq \mathsf{x} \wedge \neg \mathsf{z} < \mathsf{x} \to \mathsf{y} = \mathsf{x}^2$$

(c) The variant expression is $\mathsf{x} - \mathsf{z}$. To prove termination we need to prove two facts:

1. If the loop condition is true, then the variant expression is strictly positive: $\mathsf{z} < \mathsf{x} \rightarrow \mathsf{x} - \mathsf{z} > 0$, which is true.
2. The value of the variant expression decreases during the execution of the loop body: if $\mathsf{x} - \mathsf{z} = e_0$ at the beginning of the loop body, then $\mathsf{x} - \mathsf{z} < e_0$ at the end. (The question doesn't ask you to provide this proof but only to explain that it is what you have to do.)

Question 6

(a) The program is:

```
if n = 0 then skip
else (
  k := 1;
  b[0] := a[0];
  while k < n do (
     b[k] := min(b[k − 1], a[k]);
     k := k + 1
  )
)
```

(b)

$$\begin{aligned} \mathsf{invariant}: &\quad (\forall i, 0 \leq i < \mathsf{k} \rightarrow \mathsf{b}[i] = \min\{\mathsf{a}[0], \ldots, \mathsf{a}[i]\}) \wedge \mathsf{k} \leq \mathsf{n} \\ \mathsf{variant}: &\quad \mathsf{n} - \mathsf{k} \end{aligned}$$