Minimal-Sum Sections

G52DOA - Derivation of Algorithms

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Sections of an Array

Let a be an array of integers of length n.
The elements of a are: $\mathrm{a}[0], \ldots, \mathrm{a}[n-1]$.
A *section* of a is a contiguous sub-array of a.
Example: some sections of $\mathrm{a} = [-1, 3, 15, -6, 4, -5]$:

$$\begin{array}{ll} [3, 15, -6] & [-6] \\ [4, -5] & [-1, 3, 15, -6, 4, -5] \end{array}$$

But $[-1, 15, -6]$ is not a section: elements not contiguous.
If i and j are two indices: $0 \leq i \leq j < n$
$S_{i,j}$ denotes the sum of the corresponding section:

$$S_{1,3} = 3 + 15 + (-6) = 12 \quad S_{3,3} = -6 \quad S_{4,5} = 4 + (-5) = -1$$
$$S_{0,5} = -1 + 3 + 15 + (-6) + 4 + (-5) = 10$$

Minimal-Sum Section

Our goal is to find the minimum value that $S_{i,j}$ can have.
minimal section is one with the smallest possible sum.
In the example above the minimal sum is:

$$S_{3,5} = -6 + 4 - 5 = -7.$$

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- One for the index j;
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The naive algorithm has three loops:
One for the index i;
One for the index j;
One to sum the section.
But we can find a clever algorithm with only one loop!

Specification of Minimal-Sum Algorithm

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We can prove the two parts of the specification separately.
We do the first, second for homework.

Minimal-Sum Section Algorithm

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\begin{array}{l}
\mathsf{k} := 1 \\
\mathsf{t} := \mathsf{a}[0] \\
\mathsf{s} := \mathsf{a}[0] \\
\text{while } \mathsf{k} \neq n \text{ do (} \\
\quad \mathsf{t} := \min(\mathsf{t} + \mathsf{a}[k], \mathsf{a}[k]); \\
\quad \mathsf{s} := \min(\mathsf{s}, \mathsf{t}); \\
\quad \mathsf{k} := \mathsf{k} + 1 \\
\text{)}
\end{array}
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where the invariant is:

$$I \equiv \forall_{0 \le i \le j < \mathsf{k}}, \mathsf{s} \le S_{i,j} \wedge \forall_{0 \le i < \mathsf{k}}, \mathsf{t} \le S_{i,\mathsf{k}-1}$$

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$\{R_3\}$	$\quad \mathsf{t} := \min(\mathsf{t} + \mathsf{a}[k], \mathsf{a}[k]);$	$\{R_2\}$
$\{R_2\}$	$\quad \mathsf{s} := \min(\mathsf{s}, \mathsf{t});$	$\{R_1\}$
$\{R_1\}$	$\quad \mathsf{k} := \mathsf{k} + 1$	$\{I\}$
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Weakest Preconditions and Implications

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To Prove: $P_0 \rightarrow I$ and $I \wedge \mathrm{k} \neq n \rightarrow R_3$.